

INVERSE ESTIMATES FOR ELLIPTIC BOUNDARY INTEGRAL OPERATORS AND THEIR APPLICATION TO THE ADAPTIVE COUPLING OF FEM AND BEM

M. AURADA, M. FEISCHL, T. FÜHRER, M. KARKULIK, J. M. MELENK, AND D. PRAETORIUS

ABSTRACT. We prove inverse-type estimates for the four classical boundary integral operators associated with the Laplace operator. These estimates are used to show convergence of an h -adaptive algorithm for the coupling of a finite element method with a boundary element method which is driven by a weighted residual error estimator.

1. INTRODUCTION

A posteriori error estimation and adaptivity have a long history in finite element methods (FEMs), which harks back at least to the late 1970s. While the early mathematical analysis (see, e.g., the monographs [AO, BS, V]) focussed on *a posteriori* error estimation, significant progress has been made in the last decade in the analysis of the adaptive FEM (AFEM) regarding provable convergence and achievable convergence rates. For linear elliptic model problems and discretizations by fixed order polynomials on shape-regular meshes, convergence and even quasi-optimal convergence rates for AFEM have been proved; we refer to [CKNS] for symmetric problems as well as to [CN] for nonsymmetric problems and the references therein.

The situation is less developed for the adaptive boundary element method (ABEM) and even worse for the adaptive coupling of FEM and BEM. For the ABEM based on first kind integral equations, [FKMP] and [G] proved very recently convergence and optimality. Specifically, [FKMP, G] studied lowest order discretizations of equations related to the simple-layer operator and hypersingular operator. We highlight that the symmetry of these two operators is an important ingredient in the optimality proofs for the ABEM in [FKMP, G]. As a first step towards a full analysis of the more complex case of the adaptive coupling of FEM and BEM, we show in the present paper convergence for Costabel's symmetric coupling.

Broadly speaking, the procedure in [FKMP, G] and the present work relies on a framework delineated for AFEM in [CKNS]. The starting point for AFEM are reliable residual type error estimators. Ideally, the residual is measured in a dual norm; for example, in the classical Laplace Dirichlet problem with numerical approximation u_h one has to evaluate $\|f + \Delta u_h\|_{H^{-1}(\Omega)}$. Since such dual norms are difficult to realize computationally, the classical residual error estimators mimic local versions of them by weighted L^2 -norms of various components of the residual. The appropriate weight is given in terms of the local mesh size function h . Returning to the example of the Laplace Dirichlet problem, these are the elementwise volume residuals $\|h(f + \Delta u_h)\|_{L^2(T)}$ and the edge/face jumps of the normal

Date: November 20, 2012.

2000 Mathematics Subject Classification. 65N30, 65N15, 65N38.

Key words and phrases. FEM-BEM coupling, a posteriori error estimate, adaptive algorithm, convergence.

derivative $\|h^{1/2}[\partial_n u_h]\|_{L^2(E)}$. In effect, the residual is measured in stronger, but h -weighted Sobolev norms. Inverse estimates are therefore a key ingredient to showing efficiency of these estimators and convergence in the context of adaptive methods. A second feature of error estimators in the FEM is the local character of the volume and edge/face residuals. This feature stems from the fact that a *differential* equation is considered. As a result, the only inverse estimates needed in the FEM are classical ones relating stronger integer order Sobolev norms to weaker ones.

Our point of departure for ABEM are computable weighted residual error indicators made available in [C2, CMS, CMPS, CS1]. Analogously to the FEM, the nonlocal nature of the norm in which to measure the residual (these are typically non-integer Sobolev norms) is accounted for by h -weighted integer order Sobolev norms; for example, $\|h^{1/2}\nabla_\Gamma(\cdot)\|_{L^2(\Gamma)}$ is taken as a proxy for $|\cdot|_{H^{1/2}(\Gamma)}$, where ∇_Γ is the surface gradient on the surface Γ . The second feature of the residual error estimators in FEM mentioned above is the local character of the residual. This feature is not present in boundary integral equations. For example, the weighted residual error estimator for Symm's integral equation involving the simple-layer operator V is $\|h^{1/2}\nabla_\Gamma(f - Vu_h)\|_{L^2(\Gamma)}$; even when restricting the integral to a single element, the nonlocal nature of V involves the numerical approximation u_h on whole surface Γ . As a result, the classical inverse estimates for spaces of piecewise polynomials, which were suitable in the FEM, are insufficient for the BEM. The appropriate inverse estimates are provided in the present paper. In this connection, the works [FKMP, G] are particularly relevant. Using similar techniques, [FKMP] considers the special case of the lowest order discretization of the simple-layer operator \mathfrak{V} , whereas the present article covers arbitrary (fixed) order conforming discretizations of all four operators. The work of [G] leads to very similar results; possibly due to the use of wavelet techniques, [G] requires $C^{1,1}$ -surfaces. Instead, our analysis relies on techniques from local elliptic regularity theory, and we may thus admit polyhedral surfaces here. We finally note that all four operators appear in our formulation of the FEM-BEM coupling.

The remainder of this work is organized as follows: Section 2 collects all necessary notations and preliminaries (Section 2.1–2.2) and proves the new inverse estimates (Theorem 1 and Corollary 2), which are the main results of this work. Our analysis relies on elliptic regularity estimates for the simple-layer potential $\tilde{\mathfrak{V}}$ (Section 2.4) and the double-layer potential $\tilde{\mathfrak{K}}$ (Section 2.5). In Section 3, we consider an adaptive algorithm for Costabel's symmetric FEM-BEM coupling, which is applied to a linear transmission problem (Section 3.1). Our discretization includes the approximation of the given data so that an implementation has to deal with discrete boundary integral operators only (Section 3.4). We therefore extend the reliable error estimator of [CS2] to include data approximation terms in Proposition 13. Adapting the concept of estimator reduction [AFLP], which has also been used for $(h - h/2)$ -type error estimators in [AFP], we prove that the usual adaptive coupling (Algorithm 14) leads to a perturbed contraction for the error estimator ϱ_ℓ and thus obtain convergence $\varrho_\ell \rightarrow 0$ as $\ell \rightarrow \infty$. Since ϱ_ℓ provides an upper bound for the Galerkin error, which unlike [AFP] does *not* rely on any saturation assumption, we thus obtain convergence of the adaptive FEM-BEM coupling (Theorem 15). A short Section 3.8 discusses the extension of these result to nonlinear transmission problems. Numerical experiments in Section 4 illustrate the convergence of the adaptive FEM-BEM coupling procedure and give empirical evidence that the optimal order of convergence is, in fact, achieved.

2. INVERSE ESTIMATES FOR INTEGRAL OPERATORS

In this section, we prove certain inverse estimates for the four classical boundary integral operators associated with the Laplace operator. Independently of our work, similar estimates for fixed order piecewise polynomials have been proved in [G] by means of wavelet-based techniques. For technical reasons, [G] assumes the boundary to be fairly smooth, namely, $C^{1,1}$. In contrast, the present analysis is based on PDE techniques and allows us to treat polygonal/polyhedral boundaries. We mention that the estimate (2.12) below has already been shown in our earlier work [FKMP] in a discrete setting for lowest order elements $\Psi_\ell \in \mathcal{P}^0(\mathcal{E}_\ell)$ and is generalized here to the case of arbitrary $\psi \in L^2(\Gamma)$.

2.1. Preliminaries & general assumptions. Let Ω be a bounded Lipschitz domain in \mathbb{R}^d , for $d \geq 2$, with polygonal/polyhedral Lipschitz boundary $\Gamma = \partial\Omega$. The exterior unit normal vector field is denoted by $\boldsymbol{\nu}$. We define $\Omega^{\text{ext}} := \mathbb{R}^d \setminus \overline{\Omega}$. Let \mathcal{E}_ℓ denote a conforming triangulation of Γ into simplices, i.e.

- \mathcal{E}_ℓ is a finite set of non-degenerate $(d-1)$ -dimensional compact surface simplices, i.e., affine images of the reference simplex $E_{\text{ref}} \subset \mathbb{R}^{d-1}$ with positive surface measure;
- $\Gamma = \bigcup_{E \in \mathcal{E}_\ell} E$, i.e., Γ is covered by \mathcal{E}_ℓ ;
- for each $E, E' \in \mathcal{E}_\ell$ with $E \neq E'$, the intersection $E \cap E'$ is either empty, or a j -dimensional simplex for $j = 0, \dots, d-2$, i.e., a joint node, or a joint edge, or a joint face, etc.

Moreover, we assume that \mathcal{E}_ℓ is κ -shape regular, i.e.,

$$\max_{E \in \mathcal{E}_\ell} \frac{\text{diam}(E)^{d-1}}{|E|} \leq \kappa < \infty \quad \text{for } d \geq 3 \quad (2.1)$$

with $|\cdot|$ the $(d-1)$ -dimensional surface measure and $\text{diam}(\cdot)$ the Euclidean diameter, whereas

$$\max_{\substack{E, E' \in \mathcal{E}_\ell \\ E \cap E' \neq \emptyset}} \frac{\text{diam}(E)}{\text{diam}(E')} \leq \kappa < \infty \quad \text{for } d = 2. \quad (2.2)$$

Note that, for $d \geq 3$, conformity of \mathcal{E}_ℓ and κ -shape regularity (2.1) also imply (2.2) (though with a different, but bounded constant $\tilde{\kappa}$). With each triangulation \mathcal{E}_ℓ , we associate the local mesh size function $h_\ell \in L^\infty(\Gamma)$ which is defined elementwise by $h_\ell|_E := h_\ell(E) := |E|^{1/(d-1)}$ for all $E \in \mathcal{E}_\ell$. We stress that κ -shape regularity of \mathcal{E}_ℓ implies $h_\ell|_E \simeq \text{diam}(E)$.

Let $\gamma \subseteq \Gamma$ be a relatively open subset of Γ . With

$$\mathcal{E}_\ell^\gamma := \mathcal{E}_\ell|_\gamma := \{E \in \mathcal{E}_\ell : E \subseteq \gamma\}, \quad (2.3)$$

we denote the restriction of \mathcal{E}_ℓ to γ . It is always assumed that γ is resolved by \mathcal{E}_ℓ , i.e., \mathcal{E}_ℓ^γ is a κ -shape regular and conforming triangulation of γ .

2.2. Sobolev spaces and boundary integral operators. In this section, we very briefly fix our notation concerning Sobolev spaces and boundary integral operators and refer the reader to the monographs [M, HW, SS, Ver84] for further details and the precise definitions.

For the boundary $\Gamma = \partial\Omega$ of the bounded Lipschitz domain Ω , we denote by $\nabla_\Gamma(\cdot)$ the surface gradient. The Sobolev space $H^1(\Gamma)$ can be defined as the completion of the Lipschitz

continuous functions on Γ with respect to the norm

$$\|u\|_{H^1(\Gamma)}^2 := \|u\|_{L^2(\Gamma)}^2 + \|\nabla_\Gamma u\|_{L^2(\Gamma)}^2.$$

We denote by $\gamma_0^{\text{int}}(\cdot)$ the interior trace operator, i.e., the $\gamma_0^{\text{int}}u$ is the restriction of a function $u \in H^1(\Omega)$ to the boundary Γ . The space $H^{1/2}(\Gamma)$ is the trace space of $H^1(\Omega)$ equipped in the standard way with the quotient norm. For relatively open subsets $\gamma \subset \Gamma$ and $s \in \{-1/2, 0, 1/2\}$, we denote by

$$\begin{aligned} H^{1/2+s}(\gamma) &= \{v|_\gamma : v \in H^{1/2+s}(\Gamma)\}, \\ \widetilde{H}^{1/2+s}(\gamma) &= \{v|_\gamma : v \in H^{1/2+s}(\Gamma), \text{supp } v \subset \bar{\gamma}\}. \end{aligned}$$

the space of all restrictions of functions to γ and endow this spaces with the corresponding quotient norms. In particular, if $v \in \widetilde{H}^{1/2+s}(\gamma)$ is extended by zero to the entire boundary Γ , then $v \in H^{1/2+s}(\Gamma)$ and $\|v\|_{\widetilde{H}^{1/2+s}(\gamma)} = \|v\|_{H^{1/2+s}(\Gamma)}$. Finally, negative order spaces

$$H^{-1/2}(\Gamma) := H^{1/2}(\Gamma)', \quad \widetilde{H}^{-(1/2+s)}(\gamma) := H^{1/2+s}(\gamma)', \quad \text{and} \quad H^{-(1/2+s)}(\gamma) := \widetilde{H}^{1/2+s}(\gamma)'$$

are defined by duality, where duality pairings $\langle \cdot, \cdot \rangle$ are understood to extend the standard L^2 -scalar product. We note the continuous inclusions

$$\widetilde{H}^{\pm(1/2+s)}(\gamma) \subseteq H^{\pm(1/2+s)}(\gamma) \quad \text{as well as} \quad \widetilde{H}^{\pm(1/2+s)}(\Gamma) = H^{\pm(1/2+s)}(\Gamma).$$

The interior conormal derivative operator $\gamma_1^{\text{int}} : H_\Delta^1(\Omega) \rightarrow H^{-1/2}(\Gamma)$, where

$$H_\Delta^1(\Omega) := \{u \in H^1(\Omega) \mid -\Delta u \in L^2(\Omega)\},$$

is defined by the first Green's formula, viz.,

$$\langle \gamma_1^{\text{int}}u, v \rangle_\Gamma = \langle \nabla u, \nabla v \rangle_\Omega - \langle -\Delta u, v \rangle_\Omega \quad \text{for all } v \in H^1(\Omega). \quad (2.4)$$

Remark. The operator γ_1^{int} generalizes the classical normal derivative operator: if $u \in H_\Delta^1(\Omega)$ is sufficiently smooth near a boundary point x_0 , then γ_1^{int} can be represented near x_0 by a function given by the pointwise defined normal derivative $\partial_\nu u$. \square

The exterior trace γ_0^{ext} and the exterior conormal derivative operator γ_1^{ext} are defined analogously to their interior counterparts. To that end, we fix a bounded Lipschitz domain $U \subset \mathbb{R}^d$ with $\bar{\Omega} \subset U$. The exterior trace operator $\gamma_0^{\text{ext}} : H^1(U \setminus \bar{\Omega}) \rightarrow H^{1/2}(\Gamma)$ is defined by restricting to Γ , and the exterior conormal derivative γ_1^{ext} is characterized by $\langle \gamma_1^{\text{ext}}u, v \rangle_\Gamma = \langle \nabla u, \nabla v \rangle_{U \setminus \bar{\Omega}} - \langle -\Delta u, v \rangle_{U \setminus \bar{\Omega}}$ for all $v \in H^1(U \setminus \bar{\Omega})$.

For a function u that admits both derivatives, we define the jumps $[\gamma_1 u] := \gamma_1^{\text{ext}}u - \gamma_1^{\text{int}}u$, as well as $[u] = \gamma_0^{\text{ext}}u - \gamma_0^{\text{int}}u$.

We denote by G the fundamental solution of the d -dimensional Laplacian

$$G(x, y) = \begin{cases} -\frac{1}{|\mathbb{S}^1|} \log |x - y|, & \text{for } d = 2, \\ +\frac{1}{|\mathbb{S}^{d-1}|} |x - y|^{-(d-2)}, & \text{for } d \geq 3, \end{cases} \quad (2.5)$$

where $|\mathbb{S}^{d-1}|$ denotes the surface measure of the Euclidean sphere in \mathbb{R}^d , e.g., $|\mathbb{S}^1| = 2\pi$ and $|\mathbb{S}^2| = 4\pi$. The classical simple-layer potential $\widetilde{\mathfrak{M}}$ and the double layer potential $\widetilde{\mathfrak{K}}$ are defined

by

$$(\tilde{\mathfrak{V}}\psi)(x) := \int_{\Gamma} G(x, y)\psi(y) d\Gamma(y), \quad (\tilde{\mathfrak{K}}v)(x) := \int_{\Gamma} \partial_{\nu(y)} G(x, y)v(y) d\Gamma(y), \quad x \in \mathbb{R}^d \setminus \Gamma;$$

here, $\partial_{\nu(y)}$ denotes the (outer) normal derivative with respect to the variable y . These pointwise defined operators can be extended to bounded linear operator with

$$\tilde{\mathfrak{V}} \in L(H^{-1/2}(\Gamma); H^1(U)) \quad \text{and} \quad \tilde{\mathfrak{K}} \in L(H^{1/2}(\Gamma); H^1(U \setminus \Gamma)). \quad (2.6)$$

It is classical that $\Delta \tilde{\mathfrak{V}}\psi = 0 = \Delta \tilde{\mathfrak{K}}v$ in $U \setminus \Gamma$ for all $\psi \in H^{-1/2}(\Gamma)$ and $v \in H^{1/2}(\Gamma)$. The simple-layer, double-layer, adjoint double-layer, and the hypersingular operators are defined as follows:

$$\mathfrak{V} = \gamma_0^{\text{int}} \tilde{\mathfrak{V}}, \quad \mathfrak{K} = \frac{1}{2} + \gamma_0^{\text{int}} \tilde{\mathfrak{K}}, \quad \mathfrak{K}' = -\frac{1}{2} + \gamma_1^{\text{int}} \tilde{\mathfrak{V}}, \quad \text{and} \quad \mathfrak{W} = -\gamma_1^{\text{int}} \tilde{\mathfrak{K}}. \quad (2.7)$$

As is shown in [HW, M, Ver84], these linear operators have the following mapping properties for $s \in \{-1/2, 0, 1/2\}$ and representations (in the case of the hypersingular operator \mathfrak{W} , the integral is understood as a part finite integral):

$$\mathfrak{V} \in L(\tilde{H}^{-1/2+s}(\gamma); H^{1/2+s}(\gamma)), \quad (\mathfrak{V}\psi)(x) = \int_{\Gamma} G(x, y)\psi(y) d\Gamma(y), \quad (2.8)$$

$$\mathfrak{K} \in L(\tilde{H}^{1/2+s}(\gamma); H^{1/2+s}(\gamma)), \quad (\mathfrak{K}v)(x) = \int_{\Gamma} \partial_{\nu(y)} G(x, y)v(y) d\Gamma(y), \quad (2.9)$$

$$\mathfrak{K}' \in L(\tilde{H}^{-1/2+s}(\gamma); H^{-1/2+s}(\gamma)), \quad (\mathfrak{K}'\psi)(x) = \int_{\Gamma} \partial_{\nu(x)} G(x, y)\psi(y) d\Gamma(y), \quad (2.10)$$

$$\mathfrak{W} \in L(\tilde{H}^{1/2+s}(\gamma); H^{-1/2+s}(\gamma)), \quad (\mathfrak{W}v)(x) = -\partial_{\nu(x)} \int_{\Gamma} \partial_{\nu(y)} G(x, y)v(y) d\Gamma(y). \quad (2.11)$$

2.3. Statement of main result on inverse estimates. The following theorem is the main result of this work and the mathematical core of the arguments that allow to transfer convergence results from AFEM to ABEM.

Theorem 1. *There exists a constant $C_{\text{inv}} > 0$ such that the following estimates hold:*

$$\|h_{\ell}^{1/2} \nabla_{\Gamma} \mathfrak{V}\psi\|_{L^2(\gamma)} \leq C_{\text{inv}} (\|\psi\|_{\tilde{H}^{-1/2}(\gamma)} + \|h_{\ell}^{1/2} \psi\|_{L^2(\gamma)}), \quad (2.12)$$

$$\|h_{\ell}^{1/2} \mathfrak{K}'\psi\|_{L^2(\gamma)} \leq C_{\text{inv}} (\|\psi\|_{\tilde{H}^{-1/2}(\gamma)} + \|h_{\ell}^{1/2} \psi\|_{L^2(\gamma)}), \quad (2.13)$$

$$\|h_{\ell}^{1/2} \nabla_{\Gamma} \mathfrak{K}v\|_{L^2(\gamma)} \leq C_{\text{inv}} (\|v\|_{\tilde{H}^{1/2}(\gamma)} + \|h_{\ell}^{1/2} \nabla_{\Gamma} v\|_{L^2(\gamma)}), \quad (2.14)$$

$$\|h_{\ell}^{1/2} \mathfrak{W}v\|_{L^2(\gamma)} \leq C_{\text{inv}} (\|v\|_{\tilde{H}^{1/2}(\gamma)} + \|h_{\ell}^{1/2} \nabla_{\Gamma} v\|_{L^2(\gamma)}), \quad (2.15)$$

for all functions $\psi \in L^2(\gamma)$ and all $v \in \tilde{H}^1(\gamma)$. The constant $C_{\text{inv}} > 0$ depends only on Γ , γ , and κ -shape regularity of \mathcal{E}_{ℓ} .

Remark. The estimates (2.12)–(2.15) are rather easy to show for globally quasi-uniform meshes, i.e., $h_{\ell}(E) \simeq h_{\ell}(E')$ for all $E, E' \in \mathcal{E}_{\ell}$. For example, to see (2.12), one recalls stability of $\mathfrak{V} : L^2(\gamma) \rightarrow H^1(\gamma)$ to get

$$\|h_{\ell}^{1/2} \nabla_{\Gamma} \mathfrak{V}\psi\|_{L^2(\gamma)} \lesssim h_{\ell}^{1/2} \|\mathfrak{V}\psi\|_{H^1(\gamma)} \lesssim h_{\ell}^{1/2} \|\psi\|_{L^2(\gamma)} \simeq \|h_{\ell}^{1/2} \psi\|_{L^2(\gamma)}. \quad (2.16)$$

One sees that (2.16) is slightly stronger than (2.12), where an additional term $\|\psi\|_{\tilde{H}^{-1/2}(\gamma)}$ arises on the right-hand side. \square

For each element $E \in \mathcal{E}_\ell^\gamma$, let $\gamma_E : E_{\text{ref}} \rightarrow E$ denote the affine bijection from the reference simplex $E_{\text{ref}} \subset \mathbb{R}^{d-1}$ onto E . We introduce the space of (discontinuous) piecewise polynomials of degree q and the spaces of continuous piecewise polynomials of degree p by

$$\mathcal{P}^q(\mathcal{E}_\ell^\gamma) := \{\Psi_\ell \in L^2(\gamma) : \forall E \in \mathcal{E}_\ell^\gamma \quad \Psi_\ell \circ \gamma_E \text{ is a polynomial of degree } \leq q\}, \quad (2.17)$$

$$\mathcal{S}^p(\mathcal{E}_\ell^\gamma) := \mathcal{P}^p(\mathcal{E}_\ell^\gamma) \cap C(\gamma), \quad \text{and} \quad \mathcal{S}_0^p(\mathcal{E}_\ell^\gamma) := \{V_\ell|_\gamma : V_\ell \in \mathcal{S}^p(\mathcal{E}_\ell) \text{ with } \text{supp}(V_\ell) \subseteq \bar{\gamma}\}. \quad (2.18)$$

We note the inclusions $\mathcal{P}^q(\mathcal{E}_\ell^\gamma) \subset L^2(\gamma) \subset \tilde{H}^{-1/2}(\gamma)$, $\mathcal{S}_0^p(\mathcal{E}_\ell^\gamma) \subset \tilde{H}^1(\gamma) \subset \tilde{H}^{1/2}(\gamma)$, and $\mathcal{S}^p(\mathcal{E}_\ell^\gamma) \subseteq H^1(\gamma)$, as well as $\mathcal{S}_0^p(\mathcal{E}_\ell) = \mathcal{S}^p(\mathcal{E}_\ell)$ in case of $\gamma = \Gamma$. If we now restrict the estimates (2.12)–(2.15) of Theorem 1 to discrete functions $\Psi_\ell \in \mathcal{P}^q(\mathcal{E}_\ell^\gamma)$ and $V_\ell \in \mathcal{S}_0^p(\mathcal{E}_\ell^\gamma)$, we obtain the following estimates.

Corollary 2. *There exists a constant $\tilde{C}_{\text{inv}} > 0$ such that the following estimates hold:*

$$\max\{1, q\}^{-1} \|h_\ell^{1/2} \nabla_\Gamma \mathfrak{V} \Psi_\ell\|_{L^2(\gamma)} \leq \tilde{C}_{\text{inv}} \|\Psi_\ell\|_{\tilde{H}^{-1/2}(\gamma)}, \quad (2.19)$$

$$\max\{1, q\}^{-1} \|h_\ell^{1/2} \mathfrak{K}' \Psi_\ell\|_{L^2(\gamma)} \leq \tilde{C}_{\text{inv}} \|\Psi_\ell\|_{\tilde{H}^{-1/2}(\gamma)}, \quad (2.20)$$

$$p^{-1} \|h_\ell^{1/2} \nabla_\Gamma \mathfrak{K} V_\ell\|_{L^2(\gamma)} \leq \tilde{C}_{\text{inv}} \|V_\ell\|_{\tilde{H}^{1/2}(\gamma)}, \quad (2.21)$$

$$p^{-1} \|h_\ell^{1/2} \mathfrak{W} V_\ell\|_{L^2(\gamma)} \leq \tilde{C}_{\text{inv}} \|V_\ell\|_{\tilde{H}^{1/2}(\gamma)}, \quad (2.22)$$

for all discrete functions $\Psi_\ell \in \mathcal{P}^q(\mathcal{E}_\ell^\gamma)$ and $V_\ell \in \mathcal{S}_0^p(\mathcal{E}_\ell^\gamma)$. The constant $\tilde{C}_{\text{inv}} > 0$ depends only on γ and the shape regularity constant κ of \mathcal{E}_ℓ , but is independent of the polynomial degrees $q \geq 0$ resp. $p \geq 1$.

Remark. We stress that for discrete spaces with locally varying polynomial degrees, e.g.

$$\mathcal{P}^{q_\ell}(\mathcal{E}_\ell^\gamma) := \{\Psi_\ell \in L^2(\gamma) : \forall E \in \mathcal{E}_\ell^\gamma \quad \Psi_\ell \circ \gamma_E \text{ is a polynomial of degree } \leq q_\ell(E)\},$$

the inverse estimates of Corollary 2 remain true as long as the polynomial degrees are comparable on neighboring elements, i.e., $q_\ell(E) \simeq q_\ell(E')$ for $E \cap E' \neq \emptyset$. For example, the inverse estimate (2.19) involving \mathfrak{V} then reads

$$\|h_\ell^{1/2} \max\{1, q_\ell\}^{-1} \nabla_\Gamma \mathfrak{V} \Psi_\ell\|_{L^2(\gamma)} \leq \tilde{C}_{\text{inv}} \|\Psi_\ell\|_{\tilde{H}^{-1/2}(\gamma)} \quad \text{for all } \Psi_\ell \in \mathcal{P}^{q_\ell}(\mathcal{E}_\ell^\gamma),$$

and the estimates (2.20)–(2.22) can be extended analogously. We refer the reader to [K, Theorem 4.4], where the inverse estimates are proved in this extended fashion. \square

Proof of Corollary 2. Our starting point are two inverse estimates from [EG, Theorem 3.9] and [AKP, Proposition 3]:

$$\max\{1, q\}^{-1} \|h_\ell^{1/2} \Psi_\ell\|_{L^2(\gamma)} \lesssim \|\Psi_\ell\|_{\tilde{H}^{-1/2}(\gamma)} \quad \text{for all } \Psi_\ell \in \mathcal{P}^q(\mathcal{E}_\ell^\gamma), \quad (2.23)$$

$$p^{-1} \|h_\ell^{1/2} \nabla_\Gamma V_\ell\|_{L^2(\gamma)} \lesssim \|V_\ell\|_{\tilde{H}^{1/2}(\gamma)} \quad \text{for all } V_\ell \in \mathcal{S}_0^p(\mathcal{E}_\ell^\gamma), \quad (2.24)$$

where the hidden constants depend solely on Γ and the κ -shape regularity of \mathcal{E}_ℓ^γ . Combining (2.23) with (2.12)–(2.13) leads to (2.19)–(2.20); the bound (2.24) in conjunction with (2.14)–(2.15) yields (2.21)–(2.22). \square

The remainder of this section is devoted to the proof of Theorem 1. The proof will first be given for $\gamma = \Gamma$, and the general case $\gamma \subsetneq \Gamma$ is deduced from it afterwards.

On the technical side, an important difficulty of the proof of Theorem 1 arises from the fact that the boundary integral operators are nonlocal. We cope with this issue by splitting the operators into near field and far field contributions, each requiring different tools. The analysis of the near field part relies on local arguments and stability properties of the BIO. For the far field part, the key observation is that the BIOs are derived from two potentials, namely, the simple-layer potential $\tilde{\mathfrak{V}}$ and the double-layer potential $\tilde{\mathfrak{K}}$ by taking appropriate traces. Since these potentials solve elliptic equations, inverse type estimates (“Caccioppoli inequalities”) are available for them. The study of these two potentials is the topic of Sections 2.4 and 2.5. We will discuss the case of the simple-layer potential $\tilde{\mathfrak{V}}$ in greater detail first and be briefer afterwards in our treatment of the double-layer potential $\tilde{\mathfrak{K}}$, since the basic arguments are similar to those for $\tilde{\mathfrak{V}}$.

2.4. Far field and near field estimates for the simple layer potential. We start with subsection 2.4.1, where we introduce the decomposition of the simple layer potential into far field and near field. For either of this parts, we provide inverse estimates. Subsection 2.4.2 is devoted to the derivation of inverse estimates for the near field parts, whereas we deal with far field parts in subsection 2.4.3

2.4.1. Notation and decomposition into near field and far field. For each element $E \in \mathcal{E}_\ell$ and $\delta > 0$, we define the neighborhood U_E of E by

$$E \subset U_E := \bigcup_{x \in E} B_{2\delta h_\ell(E)}(x), \quad (2.25)$$

where $B_\varepsilon(x) := \{y \in \mathbb{R}^d \mid |x - y| < \varepsilon\} \subset \mathbb{R}^d$. By κ -shape regularity of \mathcal{E}_ℓ , there exist $\delta > 0$ and $M \in \mathbb{N}$ such that $\Gamma \cap U_E$ is contained in the patch $\omega_\ell(E)$ of E , i.e.,

$$\Gamma \cap U_E \subseteq \omega_\ell(E) := \bigcup \{E' \in \mathcal{E}_\ell : E' \cap E \neq \emptyset\}, \quad (2.26)$$

and that the covering $\Gamma \subset \bigcup_{E \in \mathcal{E}_\ell} U_E$ is locally finite, i.e.,

$$\#\{U_E : E \in \mathcal{E}_\ell \text{ and } x \in U_E\} \leq M \quad \text{for all } x \in \mathbb{R}^d \text{ and } \ell \in \mathbb{N}. \quad (2.27)$$

Finally, we fix a bounded domain $U \subset \mathbb{R}^d$ such that

$$U_E \subset U \quad \text{for all } E \in \mathcal{E}_\ell. \quad (2.28)$$

To deal with the nonlocality of the integral operators, we define for functions $\psi \in L^2(\Gamma)$ and $E \in \mathcal{E}_\ell$ the near field and the far field of the simple-layer potential $u_{\mathfrak{V}} = \tilde{\mathfrak{V}}\psi$ by

$$u_{\mathfrak{V},E}^{\text{near}} := \tilde{\mathfrak{V}}(\psi \chi_{\Gamma \cap U_E}) \quad \text{and} \quad u_{\mathfrak{V},E}^{\text{far}} := \tilde{\mathfrak{V}}(\psi \chi_{\Gamma \setminus U_E}), \quad (2.29)$$

where χ_ω denotes the characteristic function of the set $\omega \subset \mathbb{R}^d$. We have the obvious identity

$$u_{\mathfrak{V}} = u_{\mathfrak{V},E}^{\text{near}} + u_{\mathfrak{V},E}^{\text{far}} \quad \text{for all } E \in \mathcal{E}_\ell. \quad (2.30)$$

In our analysis, we will treat $u_{\mathfrak{V},E}^{\text{near}}$ and $u_{\mathfrak{V},E}^{\text{far}}$ separately, starting with the simpler case of $u_{\mathfrak{V},E}^{\text{near}}$.

2.4.2. *Inverse estimates for the near field part $u_{\mathfrak{V},E}^{\text{near}}$.* The near field parts of a potential can be treated with local arguments and the stability properties of the associated boundary integral operators.

Lemma 3. *There exists a constant $\tilde{C}_{\text{near}} > 0$ depending only on Γ and the κ -shape regularity of \mathcal{E}_ℓ such that for arbitrary $E \in \mathcal{E}_\ell$ and $\Psi_\ell^E \in \mathcal{P}^0(\mathcal{E}_\ell)$ with $\text{supp}(\Psi_\ell^E) \subseteq \omega_\ell(E)$ there holds*

$$\sum_{E \in \mathcal{E}_\ell} \|\nabla \tilde{\mathfrak{V}} \Psi_\ell^E\|_{L^2(U_E)}^2 \leq \tilde{C}_{\text{near}} \sum_{E \in \mathcal{E}_\ell} \|h_\ell^{1/2} \Psi_\ell^E\|_{L^2(\omega_\ell(E))}^2.$$

Proof. We fix an element $E \in \mathcal{E}_\ell$. Denoting by $\Psi_\ell^E(E')$ the value of $\Psi_\ell^E \in \mathcal{P}^0(\mathcal{E}_\ell)$ on the element E' we compute

$$(\nabla \tilde{\mathfrak{V}} \Psi_\ell^E)(x) = \sum_{E' \in \omega_\ell(E)} \Psi_\ell^E(E') \int_{E'} \nabla_x G(x, y) d\Gamma(y) \quad \text{for all } x \in \mathbb{R}^d \setminus \Gamma.$$

The number of elements E' in the patch $\omega_\ell(E)$ is bounded in terms of the shape regularity constant κ and thus

$$|(\nabla \tilde{\mathfrak{V}} \Psi_\ell^E)(x)|^2 \lesssim \sum_{E' \in \omega_\ell(E)} |\Psi_\ell^E(E')|^2 \left(\int_{E'} |\nabla_x G(x, y)| d\Gamma(y) \right)^2$$

with some E -independent constant, which depends only on κ . Since the mesh \mathcal{E}_ℓ is κ -shape regular, we can select a constant $c > 0$, which depends solely on κ , such that $U_E \subseteq B_{ch_\ell(E)}(b_{E'})$ and $E' \subseteq B_{ch_\ell(E)}(b_{E'})$ for each neighbor $E' \in \omega_\ell(E)$. We integrate over U_E and estimate the remaining integral

$$\begin{aligned} \int_{U_E} |(\nabla \tilde{\mathfrak{V}} \Psi_\ell^E)(x)|^2 dx &\lesssim \sum_{E' \in \omega_\ell(E)} |\Psi_\ell^E(E')|^2 \int_{U_E} \left(\int_{E'} |\nabla_x G(x, y)| d\Gamma(y) \right)^2 dx \\ &\lesssim \sum_{E' \in \omega_\ell(E)} |\Psi_\ell^E(E')|^2 \int_{B_{ch_\ell(E)}(b_{E'})} \left(\int_{B_{ch_\ell(E)}(b_{E'}) \cap \Gamma_{E'}} \frac{1}{|x - y|^{d-1}} d\Gamma(y) \right)^2 dx, \end{aligned}$$

where $\Gamma_{E'}$ denotes the hyperplane that is spanned by E' . Scaling arguments then yield

$$\begin{aligned} \int_{U_E} |(\nabla \tilde{\mathfrak{V}} \Psi_\ell^E)(x)|^2 dx &\lesssim \sum_{E' \in \omega_\ell(E)} |\Psi_\ell^E(E')|^2 h_\ell(E)^d \int_{B_1(0)} \left(\int_{B_1(0) \cap \mathbb{R}^{d-1}} \frac{1}{|x - y|^{d-1}} d\Gamma(y) \right)^2 dx \\ &\lesssim \sum_{E' \in \omega_\ell(E)} |\Psi_\ell^E(E')|^2 h_\ell(E)^d \lesssim \|h_\ell^{1/2} \Psi_\ell^E\|_{L^2(\omega_\ell(E))}^2. \end{aligned}$$

Summing this last estimate over all $E \in \mathcal{E}_\ell$, we conclude the proof. \square

Proposition 4 (Near field bound for $\tilde{\mathfrak{V}}$). *There is a constant $C_{\text{near}} > 0$ depending only on Γ and the κ -shape regularity of \mathcal{E}_ℓ such that the near field part $u_{\mathfrak{V},E}^{\text{near}}$ satisfies $u_{\mathfrak{V},E}^{\text{near}} \in H^1(U)$ and $\gamma_0^{\text{int}} u_{\mathfrak{V},E}^{\text{near}} \in H^1(\Gamma)$ as well as*

$$\sum_{E \in \mathcal{E}_\ell} \|h_\ell^{1/2} \nabla_\Gamma \gamma_0^{\text{int}} u_{\mathfrak{V},E}^{\text{near}}\|_{L^2(E)}^2 + \sum_{E \in \mathcal{E}_\ell} \|\nabla u_{\mathfrak{V},E}^{\text{near}}\|_{L^2(U_E)}^2 \leq C_{\text{near}} \|h_\ell^{1/2} \psi\|_{L^2(\Gamma)}^2. \quad (2.31)$$

Proof. The stability assertion $\mathfrak{V} : L^2(\Gamma) \rightarrow H^1(\Gamma)$ proved in [Ver84] gives, for each $E \in \mathcal{E}_\ell$,

$$\|\nabla_\Gamma \gamma_0^{\text{int}} u_{\mathfrak{V},E}^{\text{near}}\|_{L^2(E)} \leq \|\mathfrak{V}(\psi \chi_{U_E \cap \Gamma})\|_{H^1(\Gamma)} \lesssim \|\psi \chi_{U_E \cap \Gamma}\|_{L^2(\Gamma)} = \|\psi\|_{L^2(U_E \cap \Gamma)}.$$

Summing the last estimate over all $E \in \mathcal{E}_\ell$ and using the finite overlap property (2.27) of the set U_E , we arrive at

$$\begin{aligned} \sum_{E \in \mathcal{E}_\ell} \|h_\ell^{1/2} \nabla_\Gamma \gamma_0^{\text{int}} u_{\mathfrak{V},E}^{\text{near}}\|_{L^2(E)}^2 &= \sum_{E \in \mathcal{E}_\ell} h_\ell(E) \|\nabla_\Gamma \gamma_0^{\text{int}} u_{\mathfrak{V},E}^{\text{near}}\|_{L^2(E)}^2 \\ &\lesssim \sum_{E \in \mathcal{E}_\ell} h_\ell(E) \|\psi\|_{L^2(U_E \cap \Gamma)}^2 \simeq \|h_\ell^{1/2} \psi\|_{L^2(\Gamma)}^2, \end{aligned}$$

where all estimates depend only on the κ -shape regularity constant κ . This bounds the first term on the left-hand side of (2.31). To bound the second term, let Π_ℓ denote the $L^2(\Gamma)$ -orthogonal projection onto $\mathcal{P}^0(\mathcal{E}_\ell)$. We decompose the near field as $u_{\mathfrak{V},E}^{\text{near}} = \tilde{\mathfrak{W}}(\Pi_\ell(\psi \chi_{\Gamma \cap U_E})) + \tilde{\mathfrak{W}}((1 - \Pi_\ell)\psi \chi_{\Gamma \cap U_E})$. The condition $\text{supp}(\psi \chi_{\Gamma \cap U_E}) \subseteq \omega_\ell(E)$ implies $\text{supp}(\Pi_\ell(\psi \chi_{\Gamma \cap U_E})) \subseteq \omega_\ell(E)$ and therefore, taking $\Psi_\ell^E = \Pi_\ell(\psi \chi_{\Gamma \cap U_E})$ in Lemma 3, we conclude

$$\sum_{E \in \mathcal{E}_\ell} \|\nabla \tilde{\mathfrak{W}}(\Pi_\ell(\psi \chi_{\Gamma \cap U_E}))\|_{L^2(U_E)}^2 \lesssim \sum_{E \in \mathcal{E}_\ell} \|h_\ell^{1/2} \Pi_\ell(\psi \chi_{\Gamma \cap U_E})\|_{L^2(\omega_\ell(E))}^2 \lesssim \|h_\ell^{1/2} \psi\|_{L^2(\Gamma)}^2, \quad (2.32)$$

where we used the local L^2 -stability of Π_ℓ in the last estimate. Recalling the stability $\tilde{\mathfrak{W}} : H^{-1/2}(\Gamma) \rightarrow H^1(U)$ of (2.6) we can estimate

$$\begin{aligned} \|\nabla \tilde{\mathfrak{W}}((1 - \Pi_\ell)\psi \chi_{\Gamma \cap U_E})\|_{L^2(U_E)} &\lesssim \|\nabla \tilde{\mathfrak{W}}((1 - \Pi_\ell)\psi \chi_{\Gamma \cap U_E})\|_{L^2(U)} \\ &\lesssim \|(1 - \Pi_\ell)\psi \chi_{\Gamma \cap U_E}\|_{H^{-1/2}(\Gamma)} \end{aligned}$$

Together with a local approximation result for Π_ℓ from [CP1, Theorem 4.1], we obtain

$$\begin{aligned} \sum_{E \in \mathcal{E}_\ell} \|\nabla \tilde{\mathfrak{W}}((1 - \Pi_\ell)\psi \chi_{\Gamma \cap U_E})\|_{L^2(U_E)}^2 &\lesssim \sum_{E \in \mathcal{E}_\ell} \|(1 - \Pi_\ell)\psi \chi_{\Gamma \cap U_E}\|_{H^{-1/2}(\Gamma)}^2 \\ &\lesssim \sum_{E \in \mathcal{E}_\ell} \|h_\ell^{1/2}(\psi \chi_{\Gamma \cap U_E})\|_{L^2(\Gamma)}^2 \simeq \|h_\ell^{1/2} \psi\|_{L^2(\Gamma)}^2. \end{aligned} \quad (2.33)$$

The combination of (2.32)–(2.33) yields the desired estimate in (2.31) for $\sum_{E \in \mathcal{E}_\ell} \|\nabla u_{\mathfrak{V},E}^{\text{near}}\|_{L^2(U_E)}^2$. \square

2.4.3. Estimates for the far field part $u_{\mathfrak{V},E}^{\text{far}}$. The following lemma is taken from [FKMP]. For the convenience of the reader and since the same argument underlies the proof of the analogous lemma regarding the double-layer potential, we recall its proof here.

Lemma 5 (Caccioppoli inequality for $u_{\mathfrak{V},E}^{\text{far}}$). *There is a constant $C_{\text{cacc}} > 0$ depending only on the κ -shape regularity of \mathcal{E}_ℓ such that for the function $u_{\mathfrak{V},E}^{\text{far}}$ of (2.29) the following is true: $u_{\mathfrak{V},E}^{\text{far}}|_\Omega \in C^\infty(\Omega)$, $u_{\mathfrak{V},E}^{\text{far}}|_{\Omega^{\text{ext}}} \in C^\infty(\Omega^{\text{ext}})$, and $u_{\mathfrak{V},E}^{\text{far}}|_{U_E} \in C^\infty(U_E)$ with*

$$\|D^2 u_{\mathfrak{V},E}^{\text{far}}\|_{L^2(B_{\delta h_\ell(E)}(x))} \leq C_{\text{cacc}} \frac{1}{h_\ell(E)} \|\nabla u_{\mathfrak{V},E}^{\text{far}}\|_{L^2(B_{2\delta h_\ell(E)}(x))} \quad \text{for all } x \in E \in \mathcal{E}_\ell. \quad (2.34)$$

Proof. The statements $u_{\mathfrak{V},E}^{\text{far}}|_\Omega \in C^\infty(\Omega)$ and $u_{\mathfrak{V},E}^{\text{far}}|_{\Omega^{\text{ext}}} \in C^\infty(\Omega^{\text{ext}})$ are taken from [SS, Theorem 3.1.1], and we therefore focus on the statements on $u_{\mathfrak{V},E}^{\text{far}}|_{U_E} \in C^\infty(U_E)$ and the estimate (2.34). According to [SS, Proposition 3.1.7], [SS, Theorem 3.1.16], and [SS, Theorem 3.3.1],

the function $u_{\mathfrak{V},E}^{\text{far}} \in H_{\text{loc}}^1(\mathbb{R}^d) := \{v : \mathbb{R}^d \rightarrow \mathbb{R} : v|_K \in H^1(K) \text{ for all } K \subset \mathbb{R}^d \text{ compact}\}$ solves the transmission problem

$$\begin{aligned} -\Delta u_{\mathfrak{V},E}^{\text{far}} &= 0 && \text{a.e. in } \Omega \cup \Omega^{\text{ext}} \\ [u_{\mathfrak{V},E}^{\text{far}}] &= 0 && \text{in } H^{1/2}(\Gamma) \\ [\gamma_1 u_{\mathfrak{V},E}^{\text{far}}] &= -\psi \chi_{\Gamma \setminus U_E} && \text{in } H^{-1/2}(\Gamma). \end{aligned} \quad (2.35)$$

In particular, (2.35) states that the jump of $u_{\mathfrak{V},E}^{\text{far}}$ as well as the jump of the normal derivative vanish on $\Gamma \cap U_E$. This implies that $u_{\mathfrak{V},E}$ is harmonic in U_E by the following classical argument: First, we observe that $u_{\mathfrak{V},E}^{\text{far}}$ is distributionally harmonic in U_E , since a two-fold integration by parts that uses these jump conditions shows for $v \in C_0^\infty(U_E)$ that $\langle u_{\mathfrak{V},E}^{\text{far}}, -\Delta v \rangle = 0$. Weyl's lemma (see, e.g., [MO, Theorem 2.3.1]) then implies that $u_{\mathfrak{V},E}^{\text{far}}$ is therefore strongly harmonic and $u_{\mathfrak{V},E}^{\text{far}} \in C^\infty(U_E)$.

The Caccioppoli inequality (2.34) now expresses interior regularity for elliptic problems. Indeed, [MO, Lemma 5.7.1] shows

$$\|D^2 u\|_{L^2(B_r)} \lesssim \left(\|f\|_{L^2(B_{r+h})} + \frac{1}{h} \|\nabla u\|_{L^2(B_{r+h})} + \frac{1}{h^2} \|u\|_{L^2(B_{r+h})} \right) \quad (2.36)$$

for each $u \in H^1(B_{r+h})$ such that $u \in H^2(B_r)$ and $\Delta u = f$ on B_{r+h} with balls $\overline{B}_r \subset B_{r+h}$ with radii $0 < r < r+h$ and some $f \in L^2(B_{r+h})$; the hidden constant depends solely on the spatial dimension and is independent of $r, h > 0$, and u, f . We apply (2.36) with $f = 0$ and $u = u_{\mathfrak{V},E}^{\text{far}} - c_E$, where $c_E = \frac{1}{|B_{2\delta h_\ell(E)}(x)|} \int_{B_{2\delta h_\ell(E)}(x)} u_{\mathfrak{V},E}^{\text{far}}(y) dy$. Using additionally a Poincaré inequality then leads to (2.34). \square

The nonlocal character of the operator $\tilde{\mathfrak{V}}$ is represented by the far field part. Lemma 5 allows us to show a local inverse estimate for the far field part of the simple-layer operator:

Lemma 6 (Local far field bound for $\tilde{\mathfrak{V}}$). *For all $E \in \mathcal{E}_\ell$, there holds*

$$\|h_\ell^{1/2} \nabla_\Gamma \gamma_0^{\text{int}} u_{\mathfrak{V},E}^{\text{far}}\|_{L^2(E)} \leq \|h_\ell^{1/2} \nabla u_{\mathfrak{V},E}^{\text{far}}\|_{L^2(E)} \leq C_{\text{far}} \|\nabla u_{\mathfrak{V},E}^{\text{far}}\|_{L^2(U_E)}. \quad (2.37)$$

The constant $C_{\text{far}} > 0$ depends only on Γ and the κ -shape regularity constant of \mathcal{E}_ℓ .

Proof. By Lemma 5 we have $u_{\mathfrak{V},E}^{\text{far}} \in C^\infty(U_E)$. The first estimate in (2.37) follows from the fact that, for smooth functions, the surface gradient $\nabla_\Gamma(\cdot)$ is the orthogonal projection of the gradient $\nabla(\cdot)$ onto the tangent plane, i.e., $\nabla_\Gamma \gamma_0^{\text{int}} u(x) = \nabla u(x) - (\nabla u(x) \cdot \boldsymbol{\nu}(x)) \boldsymbol{\nu}(x)$, see [Ver84].

To prove the second estimate in (2.37), we fix an $E \in \mathcal{E}_\ell$. First, we select N points $x_j \in E$, $j = 1, \dots, N$, such that

$$E \subset \bigcup_{j=1}^N B_{\delta h_\ell(E)}(x_j).$$

We may assume that the number N of points depends solely on the κ -shape regularity of the mesh. This follows by geometric considerations as detailed in [FKMP, Lemma 3.5]; essentially, one may select the points x_j from a regular grid with spacing $\frac{1}{2}\delta h_\ell(E)$ and cover E with balls of radii $\delta h_\ell(E)$ centered at these points. The centers outside E that are required for the covering are then projected into E to ensure that all points are in E .

Next, let $B_i := B_{\delta h_\ell(E)}(x_i)$ and $\widehat{B}_i := B_{2\delta h_\ell(E)}(x_i) \subset U_E$. Using a standard trace inequality and the Caccioppoli inequality (2.34) we infer for all indices i that

$$\|\nabla u_{\mathfrak{V},E}^{\text{far}}\|_{L^2(B_i \cap E)}^2 \lesssim \frac{1}{h_\ell(E)} \|\nabla u_{\mathfrak{V},E}^{\text{far}}\|_{L^2(B_i)}^2 + \|\nabla u_{\mathfrak{V},E}^{\text{far}}\|_{L^2(B_i)} \|D^2 u_{\mathfrak{V},E}^{\text{far}}\|_{L^2(B_i)} \lesssim \frac{1}{h_\ell(E)} \|\nabla u_{\mathfrak{V},E}^{\text{far}}\|_{L^2(\widehat{B}_i)}^2.$$

We use the last estimate to get

$$\begin{aligned} \|\nabla_{\Gamma} \gamma_0^{\text{int}} u_E^{\text{far}}\|_{L^2(E)}^2 &\leq \|\nabla u_{\mathfrak{V},E}^{\text{far}}\|_{L^2(E)}^2 \leq \sum_{i=1}^N \|\nabla u_{\mathfrak{V},E}^{\text{far}}\|_{L^2(B_i \cap E)}^2 \lesssim \frac{1}{h_\ell(E)} \sum_{i=1}^N \|\nabla u_{\mathfrak{V},E}^{\text{far}}\|_{L^2(\widehat{B}_i)}^2 \\ &\lesssim \frac{1}{h_\ell(E)} \|\nabla u_{\mathfrak{V},E}^{\text{far}}\|_{L^2(U_E)}^2. \end{aligned}$$

This concludes the proof of (2.37). \square

Summation of the elementwise estimates of Lemma 6 yields the following result:

Proposition 7 (Far field bound for $\widetilde{\mathfrak{V}}$). *There is a constant $C_{\text{far}} > 0$ depending only on Γ and the κ -shape regularity of \mathcal{E}_ℓ such that*

$$\sum_{E \in \mathcal{E}_\ell} \|h_\ell^{1/2} \nabla_{\Gamma} \gamma_0^{\text{int}} u_{\mathfrak{V},E}^{\text{far}}\|_{L^2(E)}^2 \leq \sum_{E \in \mathcal{E}_\ell} \|h_\ell^{1/2} \nabla u_{\mathfrak{V},E}^{\text{far}}\|_{L^2(E)}^2 \leq C_{\text{far}} \left(\|\psi\|_{H^{-1/2}(\Gamma)}^2 + \|h_\ell^{1/2} \psi\|_{L^2(\Gamma)}^2 \right).$$

Proof. We use the local far field bound (2.37) of Lemma 6 and $u_{\mathfrak{V},E}^{\text{far}} = \widetilde{\mathfrak{V}}\psi - u_{\mathfrak{V},E}^{\text{near}}$,

$$\begin{aligned} \sum_{E \in \mathcal{E}_\ell} \|h_\ell^{1/2} \nabla_{\Gamma} \gamma_0^{\text{int}} u_{\mathfrak{V},E}^{\text{far}}\|_{L^2(E)}^2 &\leq \sum_{E \in \mathcal{E}_\ell} \|h_\ell^{1/2} \nabla u_{\mathfrak{V},E}^{\text{far}}\|_{L^2(E)}^2 \lesssim \sum_{E \in \mathcal{E}_\ell} \|\nabla u_{\mathfrak{V},E}^{\text{far}}\|_{L^2(U_E)}^2 \\ &\lesssim \sum_{E \in \mathcal{E}_\ell} \|\nabla \widetilde{\mathfrak{V}}\psi\|_{L^2(U_E)}^2 + \sum_{E \in \mathcal{E}_\ell} \|\nabla u_{\mathfrak{V},E}^{\text{near}}\|_{L^2(U_E)}^2. \end{aligned} \tag{2.38}$$

The first term on the right-hand side in (2.38) is estimated by stability of $\widetilde{\mathfrak{V}}$ and the finite overlap property (2.27)

$$\sum_{E \in \mathcal{E}_\ell} \|\nabla \widetilde{\mathfrak{V}}\psi\|_{L^2(U_E)}^2 \lesssim \|\nabla \widetilde{\mathfrak{V}}\psi\|_{L^2(U)}^2 \lesssim \|\psi\|_{H^{-1/2}(\Gamma)}^2.$$

The second term in (2.38) is bounded with the aid of the near field bound (2.31). \square

2.5. Far field and near field estimates for the double layer potential. Section 2.4 studied the simple layer potential in detail. Corresponding results for the double layer potential are derived in the present section.

2.5.1. Decomposition into near field and far field. We use the notation introduced in Section 2.4.1. Additionally, in order to define the near field and far field parts for the double-layer potential, we need an appropriate cut-off function: Let \mathcal{N}_ℓ denote the set of nodes of \mathcal{E}_ℓ . For any $E \in \mathcal{E}_\ell$ define

$$\eta_E := \sum_{z \in \mathcal{N}_\ell \cap \omega_\ell(E)} \eta_z, \tag{2.39}$$

where $\eta_z \in \mathcal{S}^1(\mathcal{E}_\ell)$ denotes the hat function associated with the boundary node z , i.e., $\eta_z \in \mathcal{S}^1(\mathcal{E}_\ell)$ is characterized by the condition $\eta_z(z') = \delta_{zz'}$ for all $z' \in \mathcal{N}_\ell$, where $\delta_{zz'}$

denotes the Kronecker delta. We use the abbreviation $\widehat{\omega}_\ell(E) := \omega_\ell(\omega_\ell(E)) := \bigcup \{E' \in \mathcal{E}_\ell : E' \cap \omega_\ell(E) \neq \emptyset\}$ for the second order patch, where we recall from (2.26) that $\omega_\ell(E)$ denotes the patch of $E \in \mathcal{E}_\ell$. Note that $\widehat{\omega}_\ell(E) = \text{supp}(\eta_E)$ for the cut-off function η_E of (2.39). We note

$$\eta_E|_{\Gamma \cap U_E} = 1, \quad \eta_E|_{\Gamma \setminus \omega_\ell(E)} = 0, \quad \|\eta_E\|_{L^\infty(\Gamma)} = 1, \quad \text{and} \quad \|\nabla_\Gamma \eta_E\|_{L^\infty(\Gamma)} \simeq h_\ell(E)^{-1}, \quad (2.40)$$

where the constant involved in the last estimate depends only on the κ -shape regularity of \mathcal{E}_ℓ . For the double-layer potential $u_{\mathfrak{R}} = \widetilde{\mathfrak{K}}v$ of a density $v \in H^1(\Gamma)$ we define the near field and the far field part by

$$u_{\mathfrak{R},E}^{\text{near}} := \widetilde{\mathfrak{K}}((v - c_E)\eta_E) \quad \text{and} \quad u_{\mathfrak{R},E}^{\text{far}} := \widetilde{\mathfrak{K}}((v - c_E)(1 - \eta_E)), \quad (2.41)$$

where $c_E \in \mathbb{R}$ is a constant that will be specified below. Since $\widetilde{\mathfrak{K}}1 \equiv -1$ in Ω and $\widetilde{\mathfrak{K}}1 \equiv 0$ in Ω^{ext} , we have, for every $E \in \mathcal{E}_\ell$, the identities

$$u_{\mathfrak{R}} + c_E = u_{\mathfrak{R},E}^{\text{near}} + u_{\mathfrak{R},E}^{\text{far}} \quad \text{in } \Omega \quad \text{and} \quad u_{\mathfrak{R}} = u_{\mathfrak{R},E}^{\text{near}} + u_{\mathfrak{R},E}^{\text{far}} \quad \text{in } \Omega^{\text{ext}}. \quad (2.42)$$

2.5.2. Inverse estimates for the near field part $u_{\mathfrak{R},E}^{\text{near}}$. The proof of the near field bound for the double-layer potential needs an appropriate choice of the constants $c_E \in \mathbb{R}$ in (2.41).

Lemma 8 (Poincaré inequality on patches). *For given $w \in H^1(\Gamma)$ and $E \in \mathcal{E}_\ell$, there is a constant $c_E \in \mathbb{R}$ such that*

$$\|w - c_E\|_{L^2(\widehat{\omega}_\ell(E))} \leq C_1 \|h_\ell \nabla_\Gamma w\|_{L^2(\widehat{\omega}_\ell(E))}, \quad (2.43)$$

$$\|(w - c_E)\eta_E\|_{H^{1/2}(\Gamma)} \leq C_1 \|h_\ell^{1/2} \nabla_\Gamma w\|_{L^2(\widehat{\omega}_\ell(E))}, \quad (2.44)$$

$$\|(w - c_E)\eta_E\|_{H^1(\Gamma)} \leq C_1 \|\nabla_\Gamma w\|_{L^2(\widehat{\omega}_\ell(E))}. \quad (2.45)$$

The constant $C_1 > 0$ depends only on the κ -shape regularity constant of \mathcal{E}_ℓ and on the surface measure of Γ .

Proof. The first estimate (2.43) is established by assembling local Poincaré inequalities on $\widehat{\omega}_\ell(E)$ with the help of [DS, Theorem 7.1]. The properties of the cut-off function η_E detailed in (2.40), the estimate (2.43), and the product rule yield

$$\|\nabla_\Gamma((w - c_E)\eta_E)\|_{L^2(\Gamma)} \leq \|(w - c_E)\nabla_\Gamma \eta_E\|_{L^2(\Gamma)} + \|\eta_E \nabla_\Gamma(w - c_E)\|_{L^2(\Gamma)} \lesssim \|\nabla_\Gamma w\|_{L^2(\widehat{\omega}_\ell(E))},$$

since $\widehat{\omega}_\ell(E) = \text{supp}(\eta_E)$. Hence, we obtain with the trivial bound $h_\ell(E) \leq |\Gamma|^{1/(d-1)} \lesssim 1$

$$\|(w - c_E)\eta_E\|_{H^1(\Gamma)}^2 = \|(w - c_E)\eta_E\|_{L^2(\Gamma)}^2 + \|\nabla_\Gamma((w - c_E)\eta_E)\|_{L^2(\Gamma)}^2 \lesssim \|\nabla_\Gamma w\|_{L^2(\widehat{\omega}_\ell(E))}^2.$$

This proves (2.45). It remains to verify (2.44). To that end, we recall the interpolation inequality $\|\cdot\|_{H^{1/2}(\Gamma)}^2 \lesssim \|\cdot\|_{L^2(\Gamma)} \|\cdot\|_{H^1(\Gamma)}$ and note that $\|(w - c_E)\eta_E\|_{L^2(\Gamma)} \leq \|w - c_E\|_{L^2(\widehat{\omega}_\ell(E))}$ to get

$$\begin{aligned} \|(w - c_E)\eta_E\|_{H^{1/2}(\Gamma)} &\lesssim \|(w - c_E)\eta_E\|_{L^2(\Gamma)}^{1/2} \|(w - c_E)\eta_E\|_{H^1(\Gamma)}^{1/2} \\ &\lesssim \|h_\ell \nabla_\Gamma w\|_{L^2(\widehat{\omega}_\ell(E))}^{1/2} \|\nabla_\Gamma w\|_{L^2(\widehat{\omega}_\ell(E))}^{1/2} \\ &\simeq \|h_\ell^{1/2} \nabla_\Gamma w\|_{L^2(\widehat{\omega}_\ell(E))}, \end{aligned}$$

where the last estimate hinges on κ -shape regularity of \mathcal{E}_ℓ . □

The following lemma provides an estimate for the near field part of the double-layer potential.

Proposition 9 (Near field bound for $\tilde{\mathfrak{K}}$). *Let $v \in H^1(\Gamma)$ and consider $u_{\mathfrak{K},E}^{\text{near}}$ defined by (2.41) with the constant c_E given by Lemma 8. Then $\gamma_0^{\text{int}} u_{\mathfrak{K},E}^{\text{near}} \in H^1(\Gamma)$, $u_{\mathfrak{K},E}^{\text{near}}|_{\Omega} \in H^1(\Omega)$, and $u_{\mathfrak{K},E}^{\text{near}}|_{U \setminus \bar{\Omega}} \in H^1(U \setminus \bar{\Omega})$ with*

$$\begin{aligned} & \sum_{E \in \mathcal{E}_\ell} \left(\|h_\ell^{1/2} \nabla_\Gamma \gamma_0^{\text{int}} u_{\mathfrak{K},E}^{\text{near}}\|_{L^2(E)}^2 + \|\nabla u_{\mathfrak{K},E}^{\text{near}}\|_{L^2(U_E \cap \Omega)}^2 + \|\nabla u_{\mathfrak{K},E}^{\text{near}}\|_{L^2(U_E \cap \Omega^{\text{ext}})}^2 \right) \\ & \leq C_{\text{near}} \|h_\ell^{1/2} \nabla_\Gamma v\|_{L^2(\Gamma)}^2. \end{aligned} \quad (2.46)$$

The constant $C_{\text{near}} > 0$ depends only on Γ and the κ -shape regularity of \mathcal{E}_ℓ .

Proof. First, the trace of the double-layer potential $\gamma_0^{\text{int}} \tilde{\mathfrak{K}} = \mathfrak{K} - \frac{1}{2} : H^1(\Gamma) \rightarrow H^1(\Gamma)$ is continuous, (2.9). Taking into account (2.40) and the Poincaré-type estimate (2.45), we observe

$$\|\nabla_\Gamma \gamma_0^{\text{int}} u_{\mathfrak{K},E}^{\text{near}}\|_{L^2(E)} \leq \|\nabla_\Gamma \gamma_0^{\text{int}} u_{\mathfrak{K},E}^{\text{near}}\|_{L^2(\Gamma)} \lesssim \|(v - c_E)\eta_E\|_{H^1(\hat{\omega}_\ell(E))} \lesssim \|\nabla_\Gamma v\|_{L^2(\hat{\omega}_\ell(E))}.$$

Summation over all $E \in \mathcal{E}_\ell$ shows

$$\sum_{E \in \mathcal{E}_\ell} \|h_\ell^{1/2} \nabla_\Gamma \gamma_0^{\text{int}} u_{\mathfrak{K},E}^{\text{near}}\|_{L^2(E)}^2 \lesssim \|h_\ell^{1/2} \nabla_\Gamma v\|_{L^2(\Gamma)}^2. \quad (2.47)$$

Second, we use continuity of $\tilde{\mathfrak{K}} : H^{1/2}(\Gamma) \rightarrow H^1(U \setminus \Gamma)$ of (2.6) and get

$$\|\nabla u_{\mathfrak{K},E}^{\text{near}}\|_{L^2(U_E \cap \Omega)}^2 + \|\nabla u_{\mathfrak{K},E}^{\text{near}}\|_{L^2(U_E \cap \Omega^{\text{ext}})}^2 \lesssim \|(v - c_E)\eta_E\|_{H^{1/2}(\Gamma)}^2 \lesssim \|h_\ell^{1/2} \nabla_\Gamma v\|_{L^2(\hat{\omega}_\ell(E))}^2,$$

where we have used (2.44) in the last step. Summation over all $E \in \mathcal{E}_\ell$ gives

$$\sum_{E \in \mathcal{E}_\ell} \left(\|\nabla u_{\mathfrak{K},E}^{\text{near}}\|_{L^2(U_E \cap \Omega)}^2 + \|\nabla u_{\mathfrak{K},E}^{\text{near}}\|_{L^2(U_E \cap \Omega^{\text{ext}})}^2 \right) \lesssim \|h_\ell^{1/2} \nabla_\Gamma v\|_{L^2(\Gamma)}^2. \quad (2.48)$$

Combining (2.47)–(2.48), we conclude the proof. \square

2.5.3. Estimates for the far field part $u_{\mathfrak{K},E}^{\text{far}}$. As for the simple-layer potential, we have a Caccioppoli inequality for the double-layer potential, which underlies the analysis of the far field contribution.

Lemma 10 (Caccioppoli inequality for $u_{\mathfrak{K},E}^{\text{far}}$). *For the constant C_{cacc} of Lemma 5 the functions $u_{\mathfrak{K},E}^{\text{far}}$ of (2.41) satisfy $u_{\mathfrak{K},E}^{\text{far}}|_{\Omega} \in C^\infty(\Omega)$, $u_{\mathfrak{K},E}^{\text{far}}|_{\Omega^{\text{ext}}} \in C^\infty(\Omega^{\text{ext}})$, and $u_{\mathfrak{K},E}^{\text{far}}|_{U_E} \in C^\infty(U_E)$ with*

$$\|D^2 u_{\mathfrak{K},E}^{\text{far}}\|_{L^2(B_{\delta h_\ell(E)}(x))} \leq C_{\text{cacc}} \frac{1}{h_\ell(E)} \|\nabla u_{\mathfrak{K},E}^{\text{far}}\|_{L^2(B_{2\delta h_\ell(E)}(x))} \quad \text{for all } x \in E \in \mathcal{E}_\ell. \quad (2.49)$$

Proof. The proof is very similar to that of Lemma 5. One observes that the far field $u_{\mathfrak{K},E}^{\text{far}}$ solves the transmission problem

$$\begin{aligned} -\Delta u_{\mathfrak{K},E}^{\text{far}} &= 0 & \text{a.e. in } \Omega \cup \Omega^{\text{ext}} \\ [u_{\mathfrak{K},E}^{\text{far}}] &= (v - c_E)(1 - \eta_E) & \text{in } H^{1/2}(\Gamma) \\ [\gamma_1 u_{\mathfrak{K},E}^{\text{far}}] &= 0 & \text{in } H^{-1/2}(\Gamma). \end{aligned}$$

We note that $(1 - \eta_E)|_{\Gamma \cap U_E} = 0$ by construction of η_E in (2.39). Hence, the same reasoning as in the proof of Lemma 5 can be done to reach the conclusion (2.49). \square

Lemma 11 (Local far field bound for $\tilde{\mathfrak{K}}$). *For all $E \in \mathcal{E}_\ell$ there holds*

$$\|h_\ell^{1/2} \nabla_\Gamma \gamma_0^{\text{int}} u_{\mathfrak{K},E}^{\text{far}}\|_{L^2(E)} \leq \|h_\ell^{1/2} \nabla u_{\mathfrak{K},E}^{\text{far}}\|_{L^2(E)} \leq C_{\text{far}} \|\nabla u_{\mathfrak{K},E}^{\text{far}}\|_{L^2(U_E)}. \quad (2.50)$$

The constant $C_{\text{far}} > 0$ depends only on Γ and the κ -shape regularity constant of \mathcal{E}_ℓ .

Proof. The lemma is shown in exactly the same way as the corresponding bound for the simple layer potential \mathfrak{V} in Lemma 6 appealing to the Caccioppoli inequality (2.49) instead of (2.34). \square

Proposition 12 (Far field bound for $\tilde{\mathfrak{K}}$). *There holds*

$$\sum_{E \in \mathcal{E}_\ell} \|h_\ell^{1/2} \nabla_\Gamma \gamma_0^{\text{int}} u_{\mathfrak{K},E}^{\text{far}}\|_{L^2(E)}^2 \leq \sum_{E \in \mathcal{E}_\ell} \|h_\ell^{1/2} \nabla u_{\mathfrak{K},E}^{\text{far}}\|_{L^2(E)}^2 \leq C_{\text{far}} \left(\|h_\ell^{1/2} \nabla_\Gamma v\|_{L^2(\Gamma)}^2 + \|v\|_{H^{1/2}(\Gamma)}^2 \right).$$

The constant $C_{\text{far}} > 0$ depends only on Γ and the κ -shape regularity constant of \mathcal{E}_ℓ .

Proof. We have by Lemma 11

$$\begin{aligned} \sum_{E \in \mathcal{E}_\ell} \|h_\ell^{1/2} \nabla_\Gamma \gamma_0^{\text{int}} u_{\mathfrak{K},E}^{\text{far}}\|_{L^2(E)}^2 &\leq \sum_{E \in \mathcal{E}_\ell} \|h_\ell^{1/2} \nabla u_{\mathfrak{K},E}^{\text{far}}\|_{L^2(E)}^2 \lesssim \sum_{E \in \mathcal{E}_\ell} \|\nabla u_{\mathfrak{K},E}^{\text{far}}\|_{L^2(U_E)}^2 \\ &= \sum_{E \in \mathcal{E}_\ell} \|\nabla u_{\mathfrak{K},E}^{\text{far}}\|_{L^2(U_E \cap \Omega)}^2 + \sum_{E \in \mathcal{E}_\ell} \|\nabla u_{\mathfrak{K},E}^{\text{far}}\|_{L^2(U_E \cap \Omega^{\text{ext}})}^2. \end{aligned} \quad (2.51)$$

With the properties in (2.42) and a triangle inequality we get

$$\begin{aligned} \sum_{E \in \mathcal{E}_\ell} \|h_\ell^{1/2} \nabla u_{\mathfrak{K},E}^{\text{far}}\|_{L^2(E)}^2 &\lesssim \sum_{E \in \mathcal{E}_\ell} \left(\|\nabla \tilde{\mathfrak{K}}(v - c_E)\|_{L^2(U_E \cap \Omega)}^2 + \|\nabla \tilde{\mathfrak{K}}v\|_{L^2(U_E \cap \Omega^{\text{ext}})}^2 \right) \\ &\quad + \sum_{E \in \mathcal{E}_\ell} \left(\|\nabla u_{\mathfrak{K},E}^{\text{near}}\|_{L^2(U_E \cap \Omega)}^2 + \|\nabla u_{\mathfrak{K},E}^{\text{near}}\|_{L^2(U_E \cap \Omega^{\text{ext}})}^2 \right). \end{aligned} \quad (2.52)$$

The near field contribution is bounded by Proposition 9. Furthermore, noting $\nabla \tilde{\mathfrak{K}}c_E = \nabla(-c_E) = 0$ in Ω , we get

$$\begin{aligned} \sum_{E \in \mathcal{E}_\ell} \|h_\ell^{1/2} \nabla u_{\mathfrak{K},E}^{\text{far}}\|_{L^2(E)}^2 &\lesssim \sum_{E \in \mathcal{E}_\ell} \left(\|\nabla \tilde{\mathfrak{K}}v\|_{L^2(U_E \cap \Omega)}^2 + \|\nabla \tilde{\mathfrak{K}}v\|_{L^2(U_E \cap \Omega^{\text{ext}})}^2 \right) + \|h_\ell^{1/2} \nabla_\Gamma v\|_{L^2(\Gamma)}^2 \\ &\lesssim \|v\|_{H^{1/2}(\Gamma)}^2 + \|h_\ell^{1/2} \nabla_\Gamma v\|_{L^2(\Gamma)}^2, \end{aligned}$$

where we have used continuity of $\tilde{\mathfrak{K}}$. \square

2.6. Proof of Theorem 1 for $\gamma = \Gamma$. We are now in position to prove the inverse estimates (2.12)–(2.15) of Theorem 1.

Proof of the inverse estimate (2.12) for the simple-layer potential \mathfrak{V} and $\gamma = \Gamma$. Let $\psi \in L^2(\Gamma)$. Then,

$$\begin{aligned} \|h^{1/2} \nabla_\Gamma \mathfrak{V} \psi\|_{L^2(\Gamma)}^2 &= \sum_{E \in \mathcal{E}_\ell} \|h_\ell^{1/2} \nabla_\Gamma \mathfrak{V} \psi\|_{L^2(E)}^2 \\ &\lesssim \sum_{E \in \mathcal{E}_\ell} \|h_\ell^{1/2} \nabla_\Gamma \gamma_0^{\text{int}} u_{\mathfrak{V},E}^{\text{far}}\|_{L^2(E)}^2 + \sum_{E \in \mathcal{E}_\ell} \|h_\ell^{1/2} \nabla_\Gamma \gamma_0^{\text{int}} u_{\mathfrak{V},E}^{\text{near}}\|_{L^2(E)}^2. \end{aligned} \quad (2.53)$$

Both sums on the right-hand side can be estimated with the bounds of Propositions 4 and 7. This yields

$$\|h_\ell^{1/2} \nabla_\Gamma \mathfrak{V} \psi\|_{L^2(\Gamma)} \lesssim \|\psi\|_{H^{-1/2}(\Gamma)} + \|h_\ell^{1/2} \psi\|_{L^2(\Gamma)}.$$

and concludes the proof. \square

Proof of the inverse estimate (2.13) for the adjoint double-layer potential \mathfrak{K}' and $\gamma = \Gamma$. Let $\psi \in L^2(\Gamma)$. We split the left-hand side into near field and far field contributions to obtain

$$\|h_\ell^{1/2} \mathfrak{K}' \psi\|_{L^2(\Gamma)}^2 \lesssim \sum_{E \in \mathcal{E}_\ell} h_\ell(E) \|\mathfrak{K}'(\psi \chi_{U_E \cap \Gamma})\|_{L^2(E)}^2 + \sum_{E \in \mathcal{E}_\ell} h_\ell(E) \|\mathfrak{K}'(\psi \chi_{\Gamma \setminus U_E})\|_{L^2(E)}^2. \quad (2.54)$$

The continuity $\mathfrak{K}' : L^2(\Gamma) \rightarrow L^2(\Gamma)$ stated in (2.10) yields for the near field contribution

$$\begin{aligned} \sum_{E \in \mathcal{E}_\ell} h_\ell(E) \|\mathfrak{K}'(\psi \chi_{U_E \cap \Gamma})\|_{L^2(E)}^2 &\leq \sum_{E \in \mathcal{E}_\ell} h_\ell(E) \|\mathfrak{K}'(\psi \chi_{U_E \cap \Gamma})\|_{L^2(\Gamma)}^2 \lesssim \sum_{E \in \mathcal{E}_\ell} h_\ell(E) \|\psi\|_{L^2(U_E \cap \Gamma)}^2 \\ &\lesssim \|h_\ell^{1/2} \psi\|_{L^2(\Gamma)}^2. \end{aligned}$$

For the far field contribution, we write $u_{\mathfrak{V},E}^{\text{far}} = \widetilde{\mathfrak{V}}(\psi \chi_{\Gamma \setminus U_E})$ and note that $\mathfrak{K}' = -1/2 + \gamma_1^{\text{int}} \widetilde{\mathfrak{V}}$ and clearly $(\psi \chi_{\Gamma \setminus U_E})|_E = 0$. Therefore, on E we have $\mathfrak{K}'(\psi \chi_{\Gamma \setminus U_E}) = \gamma_1^{\text{int}} u_{\mathfrak{V},E}^{\text{far}}$. Furthermore, by the smoothness of $u_{\mathfrak{V},E}^{\text{far}}$ near E (see Lemma 5) we have $\gamma_1^{\text{int}} u_{\mathfrak{V},E}^{\text{far}} = \partial_\nu u_{\mathfrak{V},E}^{\text{far}}$ on E (cf. Remark 2.2) and get

$$\|\mathfrak{K}'(\psi \chi_{\Gamma \setminus U_E})\|_{L^2(E)} = \|\gamma_1^{\text{int}} u_{\mathfrak{V},E}^{\text{far}}\|_{L^2(E)} = \|\partial_\nu u_{\mathfrak{V},E}^{\text{far}}\|_{L^2(E)} \lesssim \|\nabla u_{\mathfrak{V},E}^{\text{far}}\|_{L^2(E)}.$$

The far field contribution in (2.54) can therefore be bounded by Proposition 7

$$\sum_{E \in \mathcal{E}_\ell} h_\ell(E) \|\mathfrak{K}'(\psi \chi_{\Gamma \setminus U_E})\|_{L^2(E)}^2 \lesssim \sum_{E \in \mathcal{E}_\ell} \|h_\ell^{1/2} \nabla u_{\mathfrak{V},E}^{\text{far}}\|_{L^2(E)}^2 \lesssim \|h_\ell^{1/2} \psi\|_{L^2(\Gamma)}^2 + \|\psi\|_{H^{-1/2}(\Gamma)}^2.$$

Altogether, this gives

$$\|h_\ell^{1/2} \mathfrak{K}' \psi\|_{L^2(\Gamma)} \lesssim \|h_\ell^{1/2} \psi\|_{L^2(\Gamma)} + \|\psi\|_{H^{-1/2}(\Gamma)}$$

and concludes the proof. \square

Proof of inverse estimate (2.14) for the double-layer potential \mathfrak{K} and $\gamma = \Gamma$. Let $v \in H^1(\Gamma)$. We recall the stability of $\mathfrak{K} = \frac{1}{2} + \gamma_0^{\text{int}} \widetilde{\mathfrak{K}} : H^1(\Gamma) \rightarrow H^1(\Gamma)$, from which we conclude $\gamma_0^{\text{int}} \widetilde{\mathfrak{K}} v \in H^1(\Gamma)$. Therefore,

$$\|h_\ell^{1/2} \nabla_\Gamma \mathfrak{K} v\|_{L^2(\Gamma)} = \|h_\ell^{1/2} \nabla_\Gamma (\frac{1}{2} + \gamma_0^{\text{int}} \widetilde{\mathfrak{K}}) v\|_{L^2(\Gamma)} \leq \frac{1}{2} \|h_\ell^{1/2} \nabla_\Gamma v\|_{L^2(\Gamma)} + \|h_\ell^{1/2} \nabla_\Gamma \gamma_0^{\text{int}} u_{\mathfrak{K}}\|_{L^2(\Gamma)} \quad (2.55)$$

with $u_{\mathfrak{K}} = \widetilde{\mathfrak{K}} v$. There holds $u_{\mathfrak{K}} + c_E = u_{\mathfrak{K},E}^{\text{near}} + u_{\mathfrak{K},E}^{\text{far}}$ in Ω , cf. (2.42). For the second term on the right-hand side in (2.55), we obtain with the constants c_E of Lemma 8 that

$$\begin{aligned} \|h_\ell^{1/2} \nabla_\Gamma \gamma_0^{\text{int}} u_{\mathfrak{K}}\|_{L^2(\Gamma)}^2 &= \sum_{E \in \mathcal{E}_\ell} h_\ell(E) \|\nabla_\Gamma \gamma_0^{\text{int}} (u_{\mathfrak{K}} + c_E)\|_{L^2(E)}^2 \\ &\lesssim \sum_{E \in \mathcal{E}_\ell} h_\ell(E) \|\nabla_\Gamma \gamma_0^{\text{int}} u_{\mathfrak{K},E}^{\text{near}}\|_{L^2(E)}^2 + \sum_{E \in \mathcal{E}_\ell} h_\ell(E) \|\nabla_\Gamma \gamma_0^{\text{int}} u_{\mathfrak{K},E}^{\text{far}}\|_{L^2(E)}^2. \end{aligned} \quad (2.56)$$

The first sum can be bounded by Proposition 9, whereas the second sum can be bounded by Proposition 12. Altogether, this yields

$$\|h_\ell^{1/2} \nabla_\Gamma \mathfrak{K} v\|_{L^2(\Gamma)} \lesssim \|v\|_{H^{1/2}(\Gamma)} + \|h_\ell^{1/2} \nabla_\Gamma v\|_{L^2(\Gamma)}$$

and concludes the proof. \square

Proof of inverse estimate (2.15) for the hypersingular integral operator \mathfrak{W} and $\gamma = \Gamma$. Let again $v \in H^1(\Gamma)$. We split the left-hand side of (2.15) into the sum over all elements $E \in \mathcal{E}_\ell$. On every element, we subtract the constants c_E from Lemma 8. Note that $\mathfrak{W}c_E = 0$. Splitting now into near field and far field yields

$$\begin{aligned} \|h_\ell^{1/2} \mathfrak{W} v\|_{L^2(\Gamma)}^2 &= \sum_{E \in \mathcal{E}_\ell} \|h_\ell^{1/2} \mathfrak{W}(v - c_E)\|_{L^2(E)}^2 \\ &\lesssim \sum_{E \in \mathcal{E}_\ell} \|h_\ell^{1/2} \mathfrak{W}((v - c_E)\eta_E)\|_{L^2(E)}^2 + \sum_{E \in \mathcal{E}_\ell} \|h_\ell^{1/2} \mathfrak{W}((v - c_E)(1 - \eta_E))\|_{L^2(E)}^2. \end{aligned} \quad (2.57)$$

The near field contribution is bounded by the stability of $\mathfrak{W} : H^1(\Gamma) \rightarrow L^2(\Gamma)$ stated in (2.11):

$$\|\mathfrak{W}((v - c_E)\eta_E)\|_{L^2(E)}^2 \lesssim \|(v - c_E)\eta_E\|_{H^1(\widehat{\omega}_\ell(E))}^2 \lesssim \|\nabla_\Gamma v\|_{L^2(\widehat{\omega}_\ell(E))}^2,$$

where we have used the Poincaré-type estimate of Lemma 8 in the last step. The sum over all elements gives

$$\sum_{E \in \mathcal{E}_\ell} \|h_\ell^{1/2} \mathfrak{W}((v - c_E)\eta_E)\|_{L^2(E)}^2 \lesssim \sum_{E \in \mathcal{E}_\ell} h_\ell(E) \|\nabla_\Gamma v\|_{L^2(\widehat{\omega}_\ell(E))}^2 \lesssim \|h_\ell^{1/2} \nabla_\Gamma v\|_{L^2(\Gamma)}^2.$$

It remains to bound the second term on the right-hand side in (2.57). In view of the support properties of η_E , the potential $u_{\mathfrak{R},E}^{\text{far}} = \widetilde{\mathfrak{K}}((v - c_E)(1 - \eta_E))$ is smooth near E (cf. Lemma 10) so that $\gamma_1^{\text{int}} u_{\mathfrak{R},E}^{\text{far}} = \partial_\nu u_{\mathfrak{R},E}^{\text{far}}$ on E . Furthermore, since $\mathfrak{W} = -\gamma_1^{\text{int}} \widetilde{\mathfrak{K}}$ we see

$$\|\mathfrak{W}((v - c_E)(1 - \eta_E))\|_{L^2(E)}^2 = \|\gamma_1^{\text{int}} u_{\mathfrak{R},E}^{\text{far}}\|_{L^2(E)}^2 = \|\partial_\nu u_{\mathfrak{R},E}^{\text{far}}\|_{L^2(E)}^2 \leq \|\nabla u_{\mathfrak{R},E}^{\text{far}}\|_{L^2(E)}^2.$$

We use Proposition 12 to conclude

$$\sum_{E \in \mathcal{E}_\ell} \|h_\ell^{1/2} \mathfrak{W}((v - c_E)(1 - \eta_E))\|_{L^2(E)}^2 \leq \sum_{E \in \mathcal{E}_\ell} \|h_\ell^{1/2} \nabla u_{\mathfrak{R},E}^{\text{far}}\|_{L^2(E)}^2 \lesssim \|h_\ell^{1/2} \nabla_\Gamma v\|_{L^2(\Gamma)}^2 + \|v\|_{H^{1/2}(\Gamma)}^2.$$

Altogether, we obtain

$$\|h_\ell^{1/2} \mathfrak{W} v\|_{L^2(\Gamma)} \lesssim \|h_\ell^{1/2} \nabla_\Gamma v\|_{L^2(\Gamma)} + \|v\|_{H^{1/2}(\Gamma)}$$

and thus conclude the proof. \square

2.7. Proof of Theorem 1 for $\gamma \subsetneq \Gamma$. Finally, it remains to prove the inverse estimates (2.12)–(2.15) of Theorem 1 for the case $\gamma \subsetneq \Gamma$. Let $\psi \in L^2(\gamma)$ and $v \in \widetilde{H}^1(\gamma)$. We define the trivial extensions $\widetilde{\psi} \in L^2(\Gamma)$ and $\widetilde{v} \in H^1(\Gamma)$ by

$$\widetilde{\psi}(x) := \begin{cases} \psi(x) & \text{if } x \in \gamma \\ 0 & \text{if } x \in \Gamma \setminus \gamma \end{cases}, \quad \widetilde{v}(x) := \begin{cases} v(x) & \text{if } x \in \gamma \\ 0 & \text{if } x \in \Gamma \setminus \gamma \end{cases}.$$

Note that $\|\psi\|_{\tilde{H}^{-1/2}(\gamma)} = \|\tilde{\psi}\|_{H^{-1/2}(\Gamma)}$ and $\|v\|_{\tilde{H}^{1/2}(\gamma)} = \|\tilde{v}\|_{H^{1/2}(\Gamma)}$. With this, we see

$$\begin{aligned} \|h_\ell^{1/2} \nabla_\Gamma \mathfrak{B} \psi\|_{L^2(\gamma)} &\leq \|h_\ell^{1/2} \nabla_\Gamma \mathfrak{B} \tilde{\psi}\|_{L^2(\Gamma)} \leq C_{\text{inv}} (\|\tilde{\psi}\|_{H^{-1/2}(\Gamma)} + \|h_\ell^{1/2} \tilde{\psi}\|_{L^2(\Gamma)}) \\ &= C_{\text{inv}} (\|\psi\|_{\tilde{H}^{-1/2}(\gamma)} + \|h_\ell^{1/2} \psi\|_{L^2(\gamma)}), \end{aligned}$$

which proves estimate (2.12). The other estimates (2.13)–(2.15) follow with the same arguments. \square

3. CONVERGENT ADAPTIVE COUPLING OF FEM AND BEM

In this section, we use the inverse estimates of Section 2 to prove convergence of an adaptive FEM-BEM coupling algorithm.

3.1. Model problem. We consider the following linear interface problem

$$-\text{div}(\mathfrak{A} \nabla u^{\text{int}}) = f \quad \text{in } \Omega^{\text{int}} := \Omega, \quad (3.1a)$$

$$-\Delta u^{\text{ext}} = 0 \quad \text{in } \Omega^{\text{ext}} := \mathbb{R}^d \setminus \overline{\Omega}, \quad (3.1b)$$

$$u^{\text{int}} - u^{\text{ext}} = u_0 \quad \text{on } \Gamma, \quad (3.1c)$$

$$(\mathfrak{A} \nabla u^{\text{int}} - \nabla u^{\text{ext}}) \cdot \boldsymbol{\nu} = \phi_0 \quad \text{on } \Gamma, \quad (3.1d)$$

$$u^{\text{ext}}(x) = \mathcal{O}(1/|x|) \quad \text{as } |x| \rightarrow \infty, \quad (3.1e)$$

see Section 3.2 below for some remarks on the radiation condition (3.1e) for 2D. Here, Ω is a bounded Lipschitz domain in \mathbb{R}^d , $d = 2, 3$, with polygonal resp. polyhedral boundary $\Gamma := \partial\Omega$ and exterior unit normal vector $\boldsymbol{\nu}$. We assume that the symmetric coefficient matrix $\mathfrak{A}(x) \in \mathbb{R}_{\text{sym}}^{d \times d}$ depends Lipschitz continuously on x and has positive and bounded smallest and largest eigenvalues

$$0 < C_{\min} \leq \lambda_{\min}(x) \leq \lambda_{\max}(x) \leq C_{\max} < \infty \quad \text{for almost all } x \in \Omega \quad (3.2)$$

and x -independent constants $C_{\max} \geq C_{\min} > 0$. The given data satisfy $f \in L^2(\Omega)$, $u_0 \in H^{1/2}(\Gamma)$, and $\phi_0 \in H^{-1/2}(\Gamma)$. As usual, (3.1) is understood in the weak sense, and the sought solutions satisfy $u^{\text{int}} \in H^1(\Omega)$ and $u^{\text{ext}} \in H_{\text{loc}}^1(\Omega^{\text{ext}}) = \{v : \Omega^{\text{ext}} \rightarrow \mathbb{R} : v \in H^1(K) \text{ for all } K \subset \Omega^{\text{ext}} \text{ compact}\}$ with $\nabla u^{\text{ext}} \in L^2(\Omega^{\text{ext}})^d$.

With the boundary integral operators from (2.8)–(2.11), Problem (3.1) is equivalently stated via the symmetric FEM-BEM coupling, cf. e.g. [CS2, Theorem 1]: Find $(u, \phi) \in \mathcal{H} := H^1(\Omega) \times H^{-1/2}(\Gamma)$ such that

$$\langle \mathfrak{A} \nabla u, \nabla v \rangle_\Omega + \langle \mathfrak{W}u + (\mathfrak{K}' - \tfrac{1}{2})\phi, v \rangle_\Gamma = \langle f, v \rangle_\Omega + \langle \phi_0 + \mathfrak{W}u_0, v \rangle_\Gamma, \quad (3.3a)$$

$$\langle \psi, \mathfrak{V}\phi - (\mathfrak{K} - \tfrac{1}{2})u \rangle_\Gamma = -\langle \psi, (\mathfrak{K} - \tfrac{1}{2})u_0 \rangle_\Gamma, \quad (3.3b)$$

for all $(v, \psi) \in \mathcal{H}$.

The link between (3.1) and (3.3) is provided by $u = u^{\text{int}}$ and $\phi = \nabla u^{\text{ext}} \cdot \boldsymbol{\nu}$. Moreover, u^{ext} is then given by the third Green's formula

$$u^{\text{ext}}(x) = \tilde{\mathfrak{K}}(u - u_0)(x) - \tilde{\mathfrak{V}}\phi(x) \quad \text{for } x \in \Omega^{\text{ext}}, \quad (3.4)$$

where the potentials $\tilde{\mathfrak{V}}$ and $\tilde{\mathfrak{K}}$ formally denote the operators \mathfrak{V} and \mathfrak{K} , but are now evaluated in Ω^{ext} instead of Γ . Note carefully that we do not use a notational difference for the function

$u \in H^1(\Omega)$ and its trace $u \in H^{1/2}(\Gamma)$, for which we compute the boundary integrals $\mathfrak{W}u$ and $(\mathfrak{K} - \frac{1}{2})u$ in (3.3).

3.2. Existence and uniqueness of solutions. Assumption (3.2) guarantees pointwise ellipticity

$$C_{\text{mon}} |v - w|^2 \leq (\mathfrak{A}(x)v - \mathfrak{A}(x)w) \cdot (v - w) \quad \text{for all } v, w \in \mathbb{R}^d \text{ and } x \in \Omega \quad (3.5)$$

with (\cdot) denoting the Euclidean scalar product on \mathbb{R}^d , as well as pointwise Lipschitz continuity

$$|\mathfrak{A}(x)v - \mathfrak{A}(x)w| \leq C_{\text{lip}} |v - w| \quad \text{for all } v, w \in \mathbb{R}^d \text{ and } x \in \Omega, \quad (3.6)$$

where $C_{\text{mon}} = C_{\text{min}}, C_{\text{lip}} = C_{\text{max}}^{1/2} > 0$ do not depend on x .

Existence and uniqueness of the solution $\mathbf{u} = (u, \phi)$ of (3.3) rely on the $H^{-1/2}(\Gamma)$ -ellipticity of the simple-layer potential \mathfrak{V} . Details are found e.g. in [CS2, AFFKMP]. In 3D, the simple-layer potential \mathfrak{V} is always elliptic, i.e.

$$\|\psi\|_{H^{-1/2}(\Gamma)}^2 \lesssim \langle \psi, \mathfrak{V}\psi \rangle_{\Gamma} \quad \text{for all } \psi \in H^{-1/2}(\Gamma). \quad (3.7)$$

To ensure ellipticity in 2D, it suffices to scale $\Omega \subset \mathbb{R}^2$ appropriately so that $\text{diam}(\Omega) < 1$.

In 2D, the radiation condition $u(x) = \mathcal{O}(1/|x|)$ either has to be relaxed to $u(x) = \mathcal{O}(\log|x|)$ as $|x| \rightarrow \infty$ or the given data must satisfy the compatibility condition $\int_{\Omega} f \, dx + \int_{\Gamma} \phi_0 \, d\Gamma = 0$. The latter implies $\int_{\Gamma} \phi \, d\Gamma = 0$ and hence the right decay (3.1e) of u^{ext} at infinity, as can be seen from (3.4), cf. e.g. [S, Section 6.6.1].

3.3. Galerkin discretization. Let $\mathcal{E}_{\ell}^{\Gamma}$ be a κ -shape regular triangulation of the coupling boundary Γ in the sense of Section 2.1. In addition, let \mathcal{T}_{ℓ} be a regular triangulation of Ω into compact and non-degenerate simplices $T \in \mathcal{T}_{\ell}$. As above, we assume that Ω as well as Γ are exactly resolved by \mathcal{T}_{ℓ} and $\mathcal{E}_{\ell}^{\Gamma}$, and κ -shape regularity of \mathcal{T}_{ℓ} is understood in the sense of

$$\sigma(\mathcal{T}_{\ell}) := \max_{T \in \mathcal{T}_{\ell}} \frac{\text{diam}(T)^d}{|T|} \leq \kappa < \infty \quad (3.8)$$

with $|\cdot|$ denoting the usual volume measure on \mathbb{R}^d . By $\mathcal{E}_{\ell}^{\Omega}$, we denote the set of facets of \mathcal{T}_{ℓ} which lie inside of Ω , but not on the coupling boundary.

For the discretization, we use conforming elements and approximate u by a continuous \mathcal{T}_{ℓ} -piecewise polynomial $U_{\ell} \in \mathcal{S}^p(\mathcal{T}_{\ell}) \subset H^1(\Omega)$ of degree $p \geq 1$. Moreover, ϕ is approximated by a (possibly discontinuous) $\mathcal{E}_{\ell}^{\Gamma}$ -piecewise polynomial $\Phi_{\ell} \in \mathcal{P}^q(\mathcal{E}_{\ell}^{\Gamma}) \subset H^{-1/2}(\Gamma)$ of degree $q \geq 0$. We stress that the usual link between p and q is $q = p - 1$, and the lowest-order case would be $p = 1$ and $q = 0$.

The discrete spaces read

$$\mathcal{X}_{\ell} := \mathcal{S}^p(\mathcal{T}_{\ell}) \times \mathcal{P}^q(\mathcal{E}_{\ell}^{\Gamma}) \subseteq H^1(\Omega) \times H^{-1/2}(\Gamma) = \mathcal{H}, \quad (3.9)$$

where the product space \mathcal{H} , equipped with the canonical norm

$$\|\mathbf{v}\| = (\|v\|_{H^1(\Omega)}^2 + \|\psi\|_{H^{-1/2}(\Gamma)}^2)^{1/2} \quad \text{for } \mathbf{v} := (v, \psi) \in \mathcal{H}, \quad (3.10)$$

becomes a Hilbert space.

The Galerkin formulation of (3.3) reads as follows: Find $\mathbf{U}_\ell^* = (U_\ell^*, \Phi_\ell^*) \in \mathcal{X}_\ell$ such that

$$\langle \mathfrak{A} \nabla U_\ell^*, \nabla V_\ell \rangle_\Omega + \langle \mathfrak{W} U_\ell^* + (\mathfrak{K}' - \tfrac{1}{2}) \Phi_\ell^*, V_\ell \rangle_\Gamma = \langle f, V_\ell \rangle_\Omega + \langle \phi_0 + \mathfrak{W} u_0, V_\ell \rangle_\Gamma, \quad (3.11a)$$

$$\langle \Psi_\ell, \mathfrak{V} \Phi_\ell^* - (\mathfrak{K} - \tfrac{1}{2}) U_\ell^* \rangle_\Gamma = -\langle \Psi_\ell, (\mathfrak{K} - \tfrac{1}{2}) u_0 \rangle_\Gamma, \quad (3.11b)$$

for all $\mathbf{V}_\ell = (V_\ell, \Psi_\ell) \in \mathcal{X}_\ell$.

Again, it is known that (3.11) admits a unique discrete solution $\mathbf{U}_\ell^* \in \mathcal{X}_\ell$ which is quasi-optimal in the sense of the Céa lemma

$$\|\mathbf{u} - \mathbf{U}_\ell^*\| \leq C_{\text{Céa}} \min_{\mathbf{V}_\ell \in \mathcal{X}_\ell} \|\mathbf{u} - \mathbf{V}_\ell\|, \quad (3.12)$$

where the constant $C_{\text{Céa}} > 0$ depends only on Ω , see [AFFKMP] resp. [AFP, Appendix] for the fact that, contrary to [CS2, Corollary 3], no additional assumption on \mathcal{T}_ℓ or \mathcal{E}_ℓ^Γ is needed.

3.4. Perturbed Galerkin discretization. The right-hand side of the discrete formulation (3.11) involves the evaluation of $\mathfrak{W} u_0$ and $\mathfrak{K} u_0$, which can hardly be performed analytically. Moreover, so-called *fast methods for boundary integral operators* usually deal with discrete functions, cf. [RS]. Therefore, we propose to approximate at least the given boundary data $u_0 \in H^{1/2}(\Gamma)$ by appropriate discrete functions and proceed analogously to [KOP]: To that end and to provide below a local measure for the approximation error, we assume additional regularity $u_0 \in H^1(\Gamma)$ and use the Scott-Zhang projection $\mathcal{J}_\ell : L^2(\Gamma) \rightarrow \mathcal{S}^p(\mathcal{E}_\ell^\Gamma)$ from [SZ] to discretize $U_{0,\ell} = \mathcal{J}_\ell u_0 \in \mathcal{S}^p(\mathcal{E}_\ell^\Gamma)$. Now, the perturbed Galerkin formulation reads as follows: Find $\mathbf{U}_\ell = (U_\ell, \Phi_\ell) \in \mathcal{X}_\ell$ such that

$$\langle \mathfrak{A} \nabla U_\ell, \nabla V_\ell \rangle_\Omega + \langle \mathfrak{W} U_\ell + (\mathfrak{K}' - \tfrac{1}{2}) \Phi_\ell, V_\ell \rangle_\Gamma = \langle f, V_\ell \rangle_\Omega + \langle \phi_0 + \mathfrak{W} U_{0,\ell}, V_\ell \rangle_\Gamma, \quad (3.13a)$$

$$\langle \Psi_\ell, \mathfrak{V} \Phi_\ell - (\mathfrak{K} - \tfrac{1}{2}) U_\ell \rangle_\Gamma = -\langle \Psi_\ell, (\mathfrak{K} - \tfrac{1}{2}) U_{0,\ell} \rangle_\Gamma, \quad (3.13b)$$

for all $\mathbf{V}_\ell = (V_\ell, \Psi_\ell) \in \mathcal{X}_\ell$. Compared to (3.11), the only difference is that (3.13) involves the approximate data $U_{0,\ell}$ instead of u_0 on the right-hand side. Consequently, the same arguments as before prove that (3.13) has a unique solution. Moreover, the benefit is that (3.13) only involves discrete boundary integral operators, i.e. matrices.

Remark. We stress that the additional regularity assumption $u_0 \in H^1(\Gamma)$ is also necessary for the well-posedness of the residual error estimator of Section 3.5. \square

3.5. A posteriori error estimate. The overall goal of this section is to provide a residual *a posteriori* error estimate (3.18) for our discretization (3.13). To that end, we assume additional regularity $u_0 \in H^1(\Gamma)$ and $\phi_0 \in L^2(\Gamma)$. Recall the notation \mathcal{T}_ℓ , \mathcal{E}_ℓ^Ω , and \mathcal{E}_ℓ^Γ from Section 3.3. We define the volume residual

$$\eta_\ell(\mathcal{T}_\ell)^2 = \sum_{T \in \mathcal{T}_\ell} \eta_\ell(T)^2, \quad \text{where} \quad \eta_\ell(T)^2 = |T|^{2/d} \|f + \text{div}(\mathfrak{A} \nabla U_\ell)\|_{L^2(T)}^2, \quad (3.14)$$

the jump contributions across interior edges

$$\eta_\ell(\mathcal{E}_\ell^\Omega)^2 = \sum_{E \in \mathcal{E}_\ell^\Omega} \eta_\ell(E)^2, \quad \text{where} \quad \eta_\ell(E)^2 = |E|^{1/(d-1)} \|[(\mathfrak{A} \nabla U_\ell) \cdot \boldsymbol{\nu}]\|_{L^2(E)}^2, \quad (3.15)$$

and the boundary contributions on the coupling boundary

$$\begin{aligned}\eta_\ell(\mathcal{E}_\ell^\Gamma)^2 &= \sum_{E \in \mathcal{E}_\ell^\Gamma} \eta_\ell(E)^2, \quad \text{where} \quad \eta_\ell(E)^2 = \eta_\ell^{(1)}(E)^2 + \eta_\ell^{(2)}(E)^2, \\ \eta_\ell^{(1)}(E)^2 &= |E|^{1/(d-1)} \|\phi_0 - (\mathfrak{A}\nabla U_\ell) \cdot \boldsymbol{\nu} + \mathfrak{W}(U_{0,\ell} - U_\ell) + (\tfrac{1}{2} - \mathfrak{K}')\Phi_\ell\|_{L^2(E)}^2, \\ \eta_\ell^{(2)}(E)^2 &= |E|^{1/(d-1)} \|\nabla_\Gamma(\mathfrak{B}\Phi_\ell - (\tfrac{1}{2} - \mathfrak{K})(U_{0,\ell} - U_\ell))\|_{L^2(E)}^2.\end{aligned}\tag{3.16}$$

Moreover, we define the data approximation term

$$\text{osc}_\ell^2 = \sum_{E \in \mathcal{E}_\ell^\Gamma} \text{osc}_\ell(E)^2, \quad \text{where} \quad \text{osc}_\ell(E)^2 = |E|^{1/(d-1)} \|\nabla_\Gamma(u_0 - U_{0,\ell})\|_{L^2(E)}^2.\tag{3.17}$$

Altogether, we thus obtain the following *a posteriori* error bound, whose local contributions are used in Section 3.7 below to steer an adaptive mesh refinement.

Proposition 13. *The error is reliably estimated by*

$$\|\mathbf{u} - \mathbf{U}_\ell\|^2 \leq C_{\text{rel}} \varrho_\ell^2 = \sum_{\tau \in \mathcal{T}_\ell \cup \mathcal{E}_\ell^\Omega \cup \mathcal{E}_\ell^\Gamma} \varrho_\ell(\tau)^2 \quad \text{with} \quad \varrho_\ell(\tau)^2 = \begin{cases} \eta_\ell(\tau)^2 & \text{for } \tau \in \mathcal{T}_\ell \cup \mathcal{E}_\ell^\Omega, \\ \eta_\ell(\tau)^2 + \text{osc}_\ell(\tau)^2 & \text{for } \tau \in \mathcal{E}_\ell^\Gamma. \end{cases}\tag{3.18}$$

The constant $C_{\text{rel}} > 0$ depends only on κ -shape regularity of \mathcal{T}_ℓ and \mathcal{E}_ℓ^Γ .

Sketch of proof. Instead of solving the non-perturbed Galerkin formulation (3.11) of the weak formulation (3.3), we solve the perturbed Galerkin formulation (3.13) in practice. Put differently, $\mathbf{U}_\ell = (U_\ell, \Phi_\ell) \in \mathcal{X}_\ell$ is the Galerkin approximation of the unique solution $\mathbf{u}_\ell = (u_\ell, \phi_\ell) \in \mathcal{H}$ of the continuous perturbed formulation

$$\langle \mathfrak{A}\nabla u_\ell, \nabla v \rangle_\Omega + \langle \mathfrak{W}u_\ell + (\mathfrak{K}' - \tfrac{1}{2})\phi_\ell, v \rangle_\Gamma = \langle f, v \rangle_\Omega + \langle \phi_0 + \mathfrak{W}U_{0,\ell}, v \rangle_\Gamma,\tag{3.19a}$$

$$\langle \psi, \mathfrak{B}\phi_\ell - (\mathfrak{K} - \tfrac{1}{2})u_\ell \rangle_\Gamma = -\langle \psi, (\mathfrak{K} - \tfrac{1}{2})U_{0,\ell} \rangle_\Gamma,\tag{3.19b}$$

for all $\mathbf{v} = (v, \psi) \in \mathcal{H}$. By stability, the error between \mathbf{u} and \mathbf{u}_ℓ is controlled by

$$\|\mathbf{u} - \mathbf{u}_\ell\| \leq \|u_0 - U_{0,\ell}\|_{H^{1/2}(\Gamma)} \lesssim \|h_\ell^{1/2} \nabla_\Gamma(u_0 - U_{0,\ell})\|_{L^2(\Gamma)} \simeq \text{osc}_\ell,\tag{3.20}$$

where the required approximation estimate for the Scott-Zhang projection can be found in [KOP, Theorem 3]. Here, $h_\ell \in \mathcal{P}^0(\mathcal{E}_\ell^\Gamma)$ denotes the local mesh size function of Section 2.1. Therefore, it only remains to prove

$$\|\mathbf{u}_\ell - \mathbf{U}_\ell\| \lesssim \eta_\ell = \left(\sum_{T \in \mathcal{T}_\ell} \eta_\ell(T)^2 + \sum_{E \in \mathcal{E}_\ell^\Omega \cup \mathcal{E}_\ell^\Gamma} \eta_\ell(E)^2 \right)^{1/2}.$$

This follows along the lines of the 2D proof presented in [CS2, Theorem 4]. The only remarkable difference is that in the 3D case the estimate

$$\|\mathfrak{B}\Phi_\ell - (\tfrac{1}{2} - \mathfrak{K})(U_{0,\ell} - U_\ell)\|_{H^{1/2}(\Gamma)} \lesssim \|h_\ell^{1/2} \nabla_\Gamma(\mathfrak{B}\Phi_\ell - (\tfrac{1}{2} - \mathfrak{K})(U_{0,\ell} - U_\ell))\|_{L^2(\Gamma)}$$

does not follow from continuity arguments as in [CS2], but from a Poincaré-type estimate provided by [CMS, Corollary 4.2]. \square

Remark. (i) For the scaling of the different contributions of ϱ_ℓ defined in (3.14)–(3.17), recall that $\text{diam}(T) \simeq |T|^{1/d}$ for a d -dimensional volume element $T \in \mathcal{T}_\ell$ and $\text{diam}(E) \simeq |E|^{1/(d-1)}$ for a $(d-1)$ -dimensional boundary element $E \in \mathcal{E}_\ell^\Gamma$ or interior facets $E \in \mathcal{E}_\ell^\Omega$. Altogether, the volume contribution (3.14) is weighted by $\text{diam}(T)$, while all other contributions (3.15)–(3.17) are weighted by $\text{diam}(E)^{1/2}$.

(ii) In [KOP, Theorem 3], the approximation estimate in (3.20) is proved for any $H^{1/2}$ -stable projection \mathcal{J}_ℓ onto $\mathcal{S}^p(\mathcal{E}_\ell^\Gamma)$. For $p = 1$, it is proved in [KPP, Theorem 6] that newest vertex bisection guarantees H^1 -stability and hence also $H^{1/2}$ -stability of the L^2 -orthogonal projection $\Pi_\ell : L^2(\Gamma) \rightarrow \mathcal{S}^1(\mathcal{E}_\ell^\Gamma)$ on the 2D manifold Γ . In the case $p = 1$, we may thus also use $U_{0,\ell} = \Pi_\ell u_0$ to discretize the given Dirichlet data, and Proposition 13 still holds accordingly.

(iii) In 2D, one may also use nodal interpolation $U_{0,\ell} = I_\ell u_0$ to discretize the data, and (3.20) holds accordingly. Details are found in [AFP, Proposition 1]. \square

3.6. Local mesh refinement. Let \mathcal{T}_ℓ be a sequence of triangulations of Ω which is obtained from a given initial triangulation \mathcal{T}_0 by successive local mesh refinement. We require the following assumptions on the local mesh refinement of the volume mesh with ℓ -independent constants $0 < \kappa < \infty$ and $0 < q < 1$:

- (T1) The triangulations \mathcal{T}_ℓ of Ω are regular in the sense of Ciarlet and κ -shape regular.
- (T2) Each refined element $T \in \mathcal{T}_\ell$ is the disjoint union of its sons $T' \in \mathcal{T}_{\ell+1}$.
- (T3) The sons $T' \in \mathcal{T}_{\ell+1}$ of a refined element $T \in \mathcal{T}_\ell$ satisfy $|T'| \leq q|T|$.
- (T4) The sons $E' \in \mathcal{E}_{\ell+1}^\Omega$ of some refined facet $E \in \mathcal{E}_\ell^\Omega$ satisfy $|E'| \leq q|E|$.

In addition, let \mathcal{E}_ℓ^Γ be a sequence of triangulations of the coupling boundary Γ which is obtained from an initial triangulation \mathcal{E}_0^Γ by successive local mesh refinement. For the local mesh refinement of the boundary mesh, we assume the following:

- (E1) The triangulations \mathcal{E}_ℓ^Γ of Γ are regular in the sense of Ciarlet and κ -shape regular.
- (E2) Each refined element $E \in \mathcal{E}_\ell^\Gamma$ is the disjoint union of its sons $E' \in \mathcal{E}_{\ell+1}^\Gamma$.
- (E3) The sons $E' \in \mathcal{E}_{\ell+1}^\Gamma$ of a refined element $E \in \mathcal{E}_\ell^\Gamma$ satisfy $|E'| \leq q|E|$.

From (T2) and (E2), it follows that the discrete spaces are nested $\mathcal{X}_\ell \subseteq \mathcal{X}_{\ell+1}$.

All of these assumptions are satisfied for the usual mesh refinement strategies, see e.g. [V]. For instance, one may use newest vertex bisection for both \mathcal{T}_ℓ and \mathcal{E}_ℓ^Γ . We refer to e.g. [St] for details on the latter algorithm, but remark that (T3)–(T4) and (E3) are then satisfied with $q = 1/2$. Moreover, we stress that 2D and 3D newest vertex bisection only leads to finitely many equivalence classes of elements $T \in \bigcup_{\ell=0}^\infty \mathcal{T}_\ell$.

In the experiments below, we let $\mathcal{E}_\ell^\Gamma = \mathcal{T}_\ell|_\Gamma$ be the induced triangulation of the coupling boundary Γ and use newest vertex bisection to refine \mathcal{T}_ℓ . In 3D, \mathcal{E}_ℓ^Γ can then equivalently be obtained by use of 2D newest vertex bisection from \mathcal{E}_0^Γ . We stress, however, that this coupling is *not* needed in theory.

3.7. Adaptive algorithm and convergence. In this section, we consider the following common adaptive algorithm, which steers the local mesh refinement by use of the local contributions of $\varrho_\ell^2 = \eta_\ell^2 + \text{osc}_\ell^2$ of Proposition 13.

Algorithm 14. INPUT: Initial meshes $(\mathcal{T}_0, \mathcal{E}_0)$ for $\ell := 0$, adaptivity parameter $\theta \in (0, 1)$.

- (i) Compute discrete solution $\mathbf{U}_\ell \in \mathcal{X}_\ell$.
- (ii) Compute refinement indicators $\varrho_\ell(\tau)$ from (3.18) for all $\tau \in \mathcal{T}_\ell \cup \mathcal{E}_\ell^\Omega \cup \mathcal{E}_\ell^\Gamma$.

(iii) Determine a set $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell \cup \mathcal{E}_\ell^\Omega \cup \mathcal{E}_\ell^\Gamma$ which satisfies the Dörfler marking criterion

$$\theta \varrho_\ell^2 \leq \sum_{T \in \mathcal{T}_\ell \cap \mathcal{M}_\ell} \varrho_\ell(T)^2 + \sum_{E \in \mathcal{E}_\ell^\Omega \cap \mathcal{M}_\ell} \varrho_\ell(E)^2 + \sum_{E \in \mathcal{E}_\ell^\Gamma \cap \mathcal{M}_\ell} \varrho_\ell(E)^2. \quad (3.21)$$

- (iv) Mark elements $T \in \mathcal{T}_\ell \cap \mathcal{M}_\ell$ and facets $E \in (\mathcal{E}_\ell^\Omega \cup \mathcal{E}_\ell^\Gamma) \cap \mathcal{M}_\ell$ for refinement.
- (v) Generate $(\mathcal{T}_{\ell+1}, \mathcal{E}_{\ell+1})$ by refinement of at least all marked elements and facets.
- (vi) Increase counter $\ell \mapsto \ell + 1$, and goto (i).

OUTPUT: Sequence of error estimators $(\varrho_\ell)_{\ell \in \mathbb{N}}$ and discrete solutions $(\mathbf{U}_\ell)_{\ell \in \mathbb{N}}$. \square

Since adaptive mesh refinement does not guarantee that the volume mesh \mathcal{T}_ℓ and the boundary mesh \mathcal{E}_ℓ^Γ become infinitely fine, convergence $\mathbf{U}_\ell \rightarrow \mathbf{u}$ is a priori unclear as $\ell \rightarrow \infty$. However, we employ the concept of *estimator reduction* from [AFLP] to prove $\varrho_\ell \rightarrow 0$ as $\ell \rightarrow \infty$. Finally, convergence $\mathbf{U}_\ell \rightarrow \mathbf{u}$ as $\ell \rightarrow \infty$ then follows from the reliability (3.18) of ϱ_ℓ . These observations are stated in the following theorem.

Theorem 15. *Algorithm 14 yields a sequence of meshes \mathcal{T}_ℓ , \mathcal{E}_ℓ , and approximations \mathbf{U}_ℓ with the following properties:*

- (i) *there are constants $0 < \rho < 1$ and $C_2 > 0$ such that the overall error estimator $\varrho_\ell^2 = \eta_\ell^2 + \text{osc}_\ell^2$ satisfies*

$$\varrho_{\ell+1}^2 \leq \rho \varrho_\ell^2 + C_2 (\|\mathbf{U}_{\ell+1} - \mathbf{U}_\ell\|^2 + \|U_{0,\ell+1} - U_{0,\ell}\|_{H^{1/2}(\Gamma)}^2) \quad \text{for all } \ell \geq 0. \quad (3.22)$$

The constant $0 < \rho < 1$ depends only on the mesh size reduction constant $0 < q < 1$ from assumptions (T3)–(T4) and (E3) and the adaptivity parameter $0 < \theta < 1$. The constant $C_2 > 0$ depends additionally on Ω , κ -shape regularity of \mathcal{T}_ℓ and \mathcal{E}_ℓ^Γ , and the polynomial degrees p, q .

- (ii) *The error indicators as well as the oscillation terms tend to zero:*

$$\lim_{\ell \rightarrow \infty} \eta_\ell = 0 = \lim_{\ell \rightarrow \infty} \text{osc}_\ell. \quad (3.23)$$

- (iii) *The adaptive coupling algorithm converges: $\lim_{\ell \rightarrow \infty} \|\mathbf{u} - \mathbf{U}_\ell\| = 0$.*

Proof of Estimate (3.22) of Theorem 15. We recall that $(a + b)^2 \leq (1 + \delta)a^2 + (1 + \delta^{-1})b^2$ for all $a, b \in \mathbb{R}$ and arbitrary $\delta > 0$.

• First, for the volume contributions (3.14) and the interior jumps (3.15), we argue as in [CKNS]. By use of the triangle inequality, we see

$$\begin{aligned} \eta_{\ell+1}(\mathcal{T}_{\ell+1})^2 &= \sum_{T' \in \mathcal{T}_{\ell+1}} |T'|^{2/d} \|f + \text{div}(\mathfrak{A} \nabla U_{\ell+1})\|_{L^2(T')}^2 \\ &\leq (1 + \delta) \sum_{T' \in \mathcal{T}_{\ell+1}} |T'|^{2/d} \|f + \text{div}(\mathfrak{A} \nabla U_\ell)\|_{L^2(T')}^2 \\ &\quad + (1 + \delta^{-1}) \sum_{T' \in \mathcal{T}_{\ell+1}} |T'|^{2/d} \|\text{div} \mathfrak{A}(\nabla U_{\ell+1} - \nabla U_\ell)\|_{L^2(T')}^2 \end{aligned} \quad (3.24)$$

The product rule gives $\text{div} \mathfrak{A}(\nabla U_{\ell+1} - \nabla U_\ell) = (\text{div} \mathfrak{A}) \cdot \nabla U_{\ell+1} + \mathfrak{A} : D^2(\nabla U_{\ell+1} - \nabla U_\ell)$ with $D^2(\cdot)$ being the Hessian matrix. Using the triangle inequality and a scaling argument, the

arguments of the second sum in (3.24) are estimated by

$$\begin{aligned}
& |T'|^{2/d} \|\operatorname{div} \mathfrak{A}(\nabla U_{\ell+1} - \nabla U_\ell)\|_{L^2(T')}^2 \\
& \lesssim |T'|^{2/d} (\|\operatorname{div} \mathfrak{A}\|_{L^\infty(T')}^2 \|\nabla(U_{\ell+1} - U_\ell)\|_{L^2(T')}^2 + \|\mathfrak{A}\|_{L^\infty(T')}^2 \|D^2(\nabla U_{\ell+1} - \nabla U_\ell)\|_{L^2(T')}^2) \\
& \lesssim \|\mathfrak{A}\|_{W^{1,\infty}(T')}^2 \|\nabla(U_{\ell+1} - U_\ell)\|_{L^2(T')}^2.
\end{aligned}$$

The first sum in (3.24) is split into non-refined and refined elements, and assumption (T3) is used for the refined elements. Recall that assumption (T2) states that each element $T \in \mathcal{T}_\ell$ is the union of its sons $T' \in \mathcal{T}_{\ell+1}$. This gives

$$\begin{aligned}
& \sum_{T' \in \mathcal{T}_{\ell+1}} |T'|^{2/d} \|f + \operatorname{div}(\mathfrak{A} \nabla U_\ell)\|_{L^2(T')}^2 \\
& \leq \sum_{T \in \mathcal{T}_\ell \cap \mathcal{T}_{\ell+1}} |T|^{2/d} \|f + \operatorname{div}(\mathfrak{A} \nabla U_\ell)\|_{L^2(T)}^2 + q^{2/d} \sum_{T \in \mathcal{T}_\ell \setminus \mathcal{T}_{\ell+1}} |T|^{2/d} \|f + \operatorname{div}(\mathfrak{A} \nabla U_\ell)\|_{L^2(T)}^2 \\
& = \eta_\ell(\mathcal{T}_\ell)^2 - (1 - q^{2/d}) \eta_\ell(\mathcal{T}_\ell \setminus \mathcal{T}_{\ell+1})^2.
\end{aligned}$$

Altogether, this gives

$$\begin{aligned}
\eta_{\ell+1}(\mathcal{T}_{\ell+1})^2 & \leq (1 + \delta) (\eta_\ell(\mathcal{T}_\ell)^2 - (1 - q^{2/d}) \eta_\ell(\mathcal{T}_\ell \setminus \mathcal{T}_{\ell+1})^2) \\
& \quad + (1 + \delta^{-1}) C \|\nabla(U_{\ell+1} - U_\ell)\|_{L^2(\Omega)}^2,
\end{aligned} \tag{3.25}$$

where $C > 0$ depends on Ω , on κ -shape regularity of $\mathcal{T}_{\ell+1}$, and the (local) $W^{1,\infty}$ -norm of \mathfrak{A} .

• Second, for the interior jumps (3.15), we argue

$$\begin{aligned}
\eta_{\ell+1}(\mathcal{E}_{\ell+1}^\Omega)^2 & = \sum_{E' \in \mathcal{E}_{\ell+1}^\Omega} |E'|^{1/(d-1)} \|\mathfrak{A} \nabla U_{\ell+1} \cdot \boldsymbol{\nu}\|_{L^2(E')}^2 \\
& \leq (1 + \delta) \sum_{E' \in \mathcal{E}_{\ell+1}^\Omega} |E'|^{1/(d-1)} \|\mathfrak{A} \nabla U_\ell \cdot \boldsymbol{\nu}\|_{L^2(E')}^2 \\
& \quad + (1 + \delta^{-1}) \sum_{E' \in \mathcal{E}_{\ell+1}^\Omega} |E'|^{1/(d-1)} \|\mathfrak{A} \nabla(U_{\ell+1} - U_\ell) \cdot \boldsymbol{\nu}\|_{L^2(E')}^2
\end{aligned} \tag{3.26}$$

The arguments of the second sum in (3.26) are estimated by use of a scaling argument, namely,

$$|E'|^{1/(d-1)} \|\mathfrak{A} \nabla(U_{\ell+1} - U_\ell) \cdot \boldsymbol{\nu}\|_{L^2(E')}^2 \lesssim \|\mathfrak{A}\|_{W^{1,\infty}(\omega_{\ell+1,E'})}^2 \|\nabla(U_{\ell+1} - U_\ell)\|_{L^2(\omega_{\ell+1,E'})}^2$$

with $\omega_{\ell+1,E'} = T'_+ \cup T'_-$ being the patch of $E' = T'_+ \cap T'_- \in \mathcal{E}_{\ell+1}^\Omega$. To treat the first sum in (3.26), the essential observation is that $\mathfrak{A} \nabla U_\ell$ is continuous across new facets $\mathcal{E}_{\ell+1}^\Omega \setminus \mathcal{E}_\ell^\Omega$ so that the respective jumps vanish. This and assumption (T4) for refined interior facets give

$$\begin{aligned}
& \sum_{E' \in \mathcal{E}_{\ell+1}^\Omega} |E'|^{1/(d-1)} \|\mathfrak{A} \nabla U_\ell \cdot \boldsymbol{\nu}\|_{L^2(E')}^2 \\
& \leq \sum_{E' \in \mathcal{E}_\ell^\Omega \cap \mathcal{E}_{\ell+1}^\Omega} |E'|^{1/(d-1)} \|\mathfrak{A} \nabla U_\ell \cdot \boldsymbol{\nu}\|_{L^2(E)}^2 + q^{1/(d-1)} \sum_{E' \in \mathcal{E}_\ell^\Omega \setminus \mathcal{E}_{\ell+1}^\Omega} |E'|^{1/(d-1)} \|\mathfrak{A} \nabla U_\ell \cdot \boldsymbol{\nu}\|_{L^2(E)}^2 \\
& = \eta_\ell(\mathcal{E}_\ell^\Omega)^2 - (1 - q^{1/(d-1)}) \eta_\ell(\mathcal{E}_\ell^\Omega \setminus \mathcal{E}_{\ell+1}^\Omega)^2.
\end{aligned}$$

Altogether, we obtain

$$\begin{aligned} \eta_{\ell+1}(\mathcal{E}_{\ell+1}^\Omega)^2 &\leq (1+\delta)(\eta_\ell(\mathcal{E}_\ell^\Omega)^2 - (1-q^{1/(d-1)})\eta_\ell(\mathcal{E}_\ell^\Omega \setminus \mathcal{E}_{\ell+1}^\Omega)^2) \\ &\quad + (1+\delta^{-1})C \|\nabla(U_{\ell+1} - U_\ell)\|_{L^2(\Omega)}^2, \end{aligned} \quad (3.27)$$

where $C > 0$ depends on κ -shape regularity of $\mathcal{T}_{\ell+1}$ and the (local) $W^{1,\infty}$ -norm of \mathfrak{A} .

• Third, we consider the first boundary contribution (3.16) of the discretization error estimator. By use of the triangle inequality, we see

$$\begin{aligned} &\|\phi_0 - (\mathfrak{A}\nabla U_{\ell+1}) \cdot \boldsymbol{\nu} + \mathfrak{W}(U_{0,\ell+1} - U_{\ell+1}) + (\tfrac{1}{2} - \mathfrak{K}')\Phi_{\ell+1}\|_{L^2(E')} \\ &\leq \|\phi_0 - (\mathfrak{A}\nabla U_\ell) \cdot \boldsymbol{\nu} + \mathfrak{W}(U_{0,\ell} - U_\ell) + (\tfrac{1}{2} - \mathfrak{K}')\Phi_\ell\|_{L^2(E')} \\ &\quad + \|\mathfrak{A}\nabla(U_{\ell+1} - U_\ell) \cdot \boldsymbol{\nu}\|_{L^2(E')} + \|\mathfrak{W}((U_{0,\ell+1} - U_{0,\ell}) - (U_{\ell+1} - U_\ell))\|_{L^2(E')} \\ &\quad + \tfrac{1}{2} \|\Phi_{\ell+1} - \Phi_\ell\|_{L^2(E')} + \|\mathfrak{K}'(\Phi_{\ell+1} - \Phi_\ell)\|_{L^2(E')} \end{aligned}$$

We sum these terms over all elements $E' \in \mathcal{E}_{\ell+1}^\Gamma$ and use assumption (E3) to see

$$\begin{aligned} &\sum_{E' \in \mathcal{E}_{\ell+1}^\Gamma} |E'|^{1/(d-1)} \|\phi_0 - (\mathfrak{A}\nabla U_\ell) \cdot \boldsymbol{\nu} + \mathfrak{W}(U_{0,\ell} - U_\ell) + (\tfrac{1}{2} - \mathfrak{K}')\Phi_\ell\|_{L^2(E')}^2 \\ &\leq \sum_{E \in \mathcal{E}_\ell^\Gamma \cap \mathcal{E}_{\ell+1}^\Gamma} |E|^{1/(d-1)} \|\phi_0 - (\mathfrak{A}\nabla U_\ell) \cdot \boldsymbol{\nu} + \mathfrak{W}(U_{0,\ell} - U_\ell) + (\tfrac{1}{2} - \mathfrak{K}')\Phi_\ell\|_{L^2(E)}^2 \\ &\quad + q^{1/(d-1)} \sum_{E \in \mathcal{E}_\ell^\Gamma \setminus \mathcal{E}_{\ell+1}^\Gamma} |E|^{1/(d-1)} \|\phi_0 - (\mathfrak{A}\nabla U_\ell) \cdot \boldsymbol{\nu} + \mathfrak{W}(U_{0,\ell} - U_\ell) + (\tfrac{1}{2} - \mathfrak{K}')\Phi_\ell\|_{L^2(E)}^2. \end{aligned}$$

A scaling argument reveals

$$\sum_{E' \in \mathcal{E}_{\ell+1}^\Gamma} |E'|^{1/(d-1)} \|\mathfrak{A}\nabla(U_{\ell+1} - U_\ell) \cdot \boldsymbol{\nu}\|_{L^2(E')}^2 \lesssim \|\nabla(U_{\ell+1} - U_\ell)\|_{L^2(\Omega)}^2,$$

where the implied constant depends on the (local) $W^{1,\infty}$ -norm of \mathfrak{A} and κ -shape regularity of $\mathcal{T}_{\ell+1}$. The inverse estimates of Corollary 2 for \mathfrak{K}' and \mathfrak{W} shows

$$\begin{aligned} &\sum_{E' \in \mathcal{E}_{\ell+1}^\Gamma} |E'|^{1/(d-1)} (\|\mathfrak{W}((U_{0,\ell+1} - U_{0,\ell}) - (U_{\ell+1} - U_\ell))\|_{L^2(E')}^2 + \|\mathfrak{K}'(\Phi_{\ell+1} - \Phi_\ell)\|_{L^2(E')}^2) \\ &= \|h_{\ell+1}^{1/2} \mathfrak{W}((U_{0,\ell+1} - U_{0,\ell}) - (U_{\ell+1} - U_\ell))\|_{L^2(\Gamma)}^2 + \|h_{\ell+1}^{1/2} \mathfrak{K}'(\Phi_{\ell+1} - \Phi_\ell)\|_{L^2(\Gamma)}^2 \\ &\lesssim \|(U_{0,\ell+1} - U_{0,\ell}) - (U_{\ell+1} - U_\ell)\|_{H^{1/2}(\Gamma)}^2 + \|\Phi_{\ell+1} - \Phi_\ell\|_{H^{-1/2}(\Gamma)}^2. \end{aligned}$$

An inverse estimate from [GHS, Theorem 3.6] proves

$$\sum_{E' \in \mathcal{E}_{\ell+1}^\Gamma} |E'|^{1/(d-1)} \|\Phi_{\ell+1} - \Phi_\ell\|_{L^2(E')} = \|h_{\ell+1}^{1/2}(\Phi_{\ell+1} - \Phi_\ell)\|_{L^2(\Gamma)} \lesssim \|\Phi_{\ell+1} - \Phi_\ell\|_{H^{-1/2}(\Gamma)}^2,$$

where the implied constant depends only on Γ , the κ -shape regularity of $\mathcal{E}_{\ell+1}^\Gamma$, and the polynomial degree q . Combining all these estimates, we obtain

$$\begin{aligned} \eta_{\ell+1}^{(1)}(\mathcal{E}_{\ell+1}^\Gamma)^2 &\leq (1+\delta)(\eta_\ell^{(1)}(\mathcal{E}_\ell^\Gamma)^2 - (1-q^{1/(d-1)})\eta_\ell^{(1)}(\mathcal{E}_\ell^\Gamma \setminus \mathcal{E}_{\ell+1}^\Gamma)^2) \\ &\quad + (1+\delta^{-1})C(\|\nabla(U_{\ell+1}-U_\ell)\|_{L^2(\Omega)}^2 + \|U_{\ell+1}-U_\ell\|_{H^{1/2}(\Gamma)}^2 \\ &\quad + \|U_{0,\ell+1}-U_{0,\ell}\|_{H^{1/2}(\Gamma)}^2 + \|\Phi_{\ell+1}-\Phi_\ell\|_{H^{-1/2}(\Gamma)}^2) \end{aligned} \quad (3.28)$$

for the first contribution to $\eta_\ell(\mathcal{E}_\ell^\Gamma)$ from (3.16). The constant $C > 0$ depends only on Γ , on κ -shape regularity of $\mathcal{E}_{\ell+1}^\Gamma$ and $\mathcal{T}_{\ell+1}$, and on the polynomial degree q .

• Fourth, we consider the second boundary contribution (3.16) of the discretization error estimator. Arguing as before, we now use the inverse estimates from Corollary 2 for \mathfrak{V} and \mathfrak{K} as well as a local inverse estimate $\|h_\ell \nabla_\Gamma V_\ell\|_{L^2(\Gamma)} \lesssim \|V_\ell\|_{H^{1/2}(\Gamma)}$ for all $V_\ell \in \mathcal{S}^p(\mathcal{T}_\ell)$ which is e.g. found in [CP2, Proposition 4.1] for 2D and [AKP, Proposition 3] for 3D. We obtain

$$\begin{aligned} \eta_{\ell+1}^{(2)}(\mathcal{E}_{\ell+1}^\Gamma)^2 &\leq (1+\delta)(\eta_\ell^{(2)}(\mathcal{E}_\ell^\Gamma)^2 - (1-q^{1/(d-1)})\eta_\ell^{(2)}(\mathcal{E}_\ell^\Gamma \setminus \mathcal{E}_{\ell+1}^\Gamma)^2) \\ &\quad + (1+\delta^{-1})C(\|U_{\ell+1}-U_\ell\|_{H^{1/2}(\Gamma)}^2 \\ &\quad + \|U_{0,\ell+1}-U_{0,\ell}\|_{H^{1/2}(\Gamma)}^2 + \|\Phi_{\ell+1}-\Phi_\ell\|_{H^{-1/2}(\Gamma)}^2) \end{aligned} \quad (3.29)$$

for the first contribution to $\eta_\ell(\mathcal{E}_\ell^\Gamma)$ from (3.16). The constant $C > 0$ depends only on κ -shape regularity of $\mathcal{E}_{\ell+1}^\Gamma$ and $\mathcal{T}_{\ell+1}$, on Γ , and on the polynomial degree p .

• Fifth, we consider the oscillation terms from (3.17). Arguing as in the previous steps, it follows that

$$\begin{aligned} \text{osc}_{\ell+1}^2 &\leq (1+\delta)(\text{osc}_\ell^2 - (1-q^{1/(d-1)})\text{osc}_\ell(\mathcal{E}_\ell^\Gamma \setminus \mathcal{E}_{\ell+1}^\Gamma)^2) \\ &\quad + (1+\delta^{-1})C\|U_{0,\ell+1}-U_{0,\ell}\|_{H^{1/2}(\Gamma)}^2, \end{aligned} \quad (3.30)$$

where the constant $C > 0$ depends on Γ , κ -shape regularity of \mathcal{E}_ℓ^Γ , and the polynomial degree p .

• Sixth, we combine the reduction estimates (3.25), (3.27), (3.28), (3.29), (3.30) for the different parts of ϱ_ℓ . By a norm equivalence (with constants depending only on Ω) we have $\|\nabla(U_{\ell+1}-U_\ell)\|_{L^2(\Omega)}^2 + \|U_{\ell+1}-U_\ell\|_{H^{1/2}(\Gamma)}^2 \simeq \|U_{\ell+1}-U_\ell\|_{H^1(\Omega)}^2$. Together with $0 < q < 1$ and hence $q^{2/d} \leq q^{1/(d-1)}$, we obtain

$$\begin{aligned} \varrho_{\ell+1}^2 &= \eta_{\ell+1}(\mathcal{T}_{\ell+1})^2 + \eta_{\ell+1}(\mathcal{E}_{\ell+1}^\Omega)^2 + \eta_{\ell+1}(\mathcal{E}_{\ell+1}^\Gamma)^2 + \text{osc}_{\ell+1}^2 \\ &\leq (1+\delta)\left(\eta_\ell(\mathcal{T}_\ell)^2 + \eta_\ell(\mathcal{E}_\ell^\Omega)^2 + \eta_\ell(\mathcal{E}_\ell^\Gamma)^2 + \text{osc}_\ell^2 \right. \\ &\quad \left. - (1-q^{1/(d-1)})(\eta_\ell(\mathcal{T}_\ell \setminus \mathcal{T}_{\ell+1})^2 + \eta_\ell(\mathcal{E}_\ell^\Omega \setminus \mathcal{E}_{\ell+1}^\Omega)^2 + \eta_\ell(\mathcal{E}_\ell^\Gamma \setminus \mathcal{E}_{\ell+1}^\Gamma)^2 + \text{osc}_\ell(\mathcal{E}_\ell^\Gamma \setminus \mathcal{E}_{\ell+1}^\Gamma)^2)\right) \\ &\quad + (1+\delta^{-1})C\left(\|U_{\ell+1}-U_\ell\|_{H^1(\Omega)}^2 + \|U_{0,\ell+1}-U_{0,\ell}\|_{H^{1/2}(\Gamma)}^2 + \|\Phi_{\ell+1}-\Phi_\ell\|_{H^{-1/2}(\Gamma)}^2\right), \end{aligned}$$

and the constant $C > 0$ depends only on Ω , κ -shape regularity of \mathcal{T}_ℓ and \mathcal{E}_ℓ^Γ , and the polynomial degrees p, q . To abbreviate notation, we define the index set $\mathcal{I}_\ell := \mathcal{T}_\ell \cup \mathcal{E}_\ell^\Omega \cup \mathcal{E}_\ell^\Gamma$. Together with norm equivalence $\|U_{\ell+1}-U_\ell\|_{H^1(\Omega)}^2 + \|\Phi_{\ell+1}-\Phi_\ell\|_{H^{-1/2}(\Gamma)}^2 \simeq \|\mathbf{U}_{\ell+1}-\mathbf{U}_\ell\|^2$, the previous estimate takes the form

$$\begin{aligned} \varrho_{\ell+1}^2 &\leq (1+\delta)(\varrho_\ell^2 - (1-q^{1/(d-1)})\varrho_\ell(\mathcal{I}_\ell \setminus \mathcal{I}_{\ell+1})^2) \\ &\quad + (1+\delta^{-1})C(\|\mathbf{U}_{\ell+1}-\mathbf{U}_\ell\|^2 + \|U_{0,\ell+1}-U_{0,\ell}\|_{H^{1/2}(\Gamma)}^2). \end{aligned} \quad (3.31)$$

• Seventh, recall that marked elements and facets are refined so that $\mathcal{M}_\ell \subseteq \mathcal{I}_\ell \setminus \mathcal{I}_{\ell+1}$. With the abbreviate notation, the Dörfler marking (3.21) takes the form

$$\theta \varrho_\ell^2 \leq \varrho_\ell(\mathcal{M}_\ell)^2 \leq \varrho_\ell(\mathcal{I}_\ell \setminus \mathcal{I}_{\ell+1})^2.$$

Therefore, the estimate (3.31) becomes

$$\varrho_{\ell+1}^2 = (1 + \delta)(1 - (1 - q^{1/(d-1)})\theta) \varrho_\ell^2 + (1 + \delta^{-1})C(\|\mathbf{U}_{\ell+1} - \mathbf{U}_\ell\|^2 + \|U_{0,\ell+1} - U_{0,\ell}\|_{H^{1/2}(\Gamma)}^2).$$

Now, note $0 < (1 - q^{1/(d-1)})\theta < 1$. Therefore, we may choose some sufficiently small $\delta > 0$ such that $\rho := (1 + \delta)(1 - (1 - q^{1/(d-1)})\theta) < 1$. This concludes the proof of the estimate (3.22). \square

We recall that assumptions (T2) and (E2) imply $\mathcal{X}_\ell \subseteq \mathcal{X}_{\ell+1}$. The proof of convergence (3.23) of the adaptive coupling relies on the following a priori convergence result from [AFP, Proof of Proposition 10] which essentially follows from the nestedness $\mathcal{X}_\ell \subseteq \mathcal{X}_{\ell+1}$ of discrete spaces and the validity of C ea's lemma (3.12).

Lemma 16. *According to assumption (T2) and (E2), the limit $\mathbf{U}_\infty^* := \lim_\ell \mathbf{U}_\ell^*$ of Galerkin solutions exists strongly in \mathcal{H} . If the limit $U_{0,\infty} := \lim_\ell U_{0,\ell}$ exists strongly in $H^{1/2}(\Gamma)$, then also the limit $\mathbf{U}_\infty := \lim_\ell \mathbf{U}_\ell$ of the perturbed Galerkin solutions exists strongly in \mathcal{H} . \square*

Moreover, the following result from [FPP, Lemma 18] proves a priori convergence of $U_{0,\ell}$ for the Scott-Zhang projection. We remark that [FPP, lemma 18] is stated and proved for $p = 1$ and $H^1(\Omega)$, but the proof holds without any modification also for general $p \geq 1$ and $H^1(\Gamma)$. We recall that assumption (E2) implies nestedness $\mathcal{S}^p(\mathcal{E}_\ell^\Gamma) \subseteq \mathcal{S}^p(\mathcal{E}_{\ell+1}^\Gamma)$, whereas (E1) states uniform κ -shape regularity which enters the approximation and stability estimates of the Scott-Zhang projection.

Lemma 17. *According to (E1) and (E2), the limit $\mathcal{J}_\infty g = \lim_\ell \mathcal{J}_\ell g$ of the Scott-Zhang projections exists strongly in $H^1(\Gamma)$ for all $g \in H^1(\Gamma)$. \square*

Proof of Convergence (3.23) of Theorem 15. According to Lemma 17, the limit $U_{0,\infty} = \lim_\ell U_{0,\ell}$ exists strongly in $H^1(\Gamma)$ and hence also in $H^{1/2}(\Gamma)$. Therefore, Lemma 16 applies and proves that $\mathbf{U}_\infty := \lim_\ell \mathbf{U}_\ell$ exists in \mathcal{H} . Consequently, the estimator reduction estimate (3.22) may be written as

$$\varrho_{\ell+1}^2 \leq \rho \varrho_\ell^2 + \alpha_\ell \quad \text{for all } \ell \geq 0$$

with a non-negative zero sequence $\alpha_\ell \rightarrow 0$ as $\ell \rightarrow \infty$. It is thus a consequence of elementary calculus that $\lim_\ell \varrho_\ell = 0$, cf. [AFLP, Lemma 2.3]. With $\varrho_\ell^2 = \eta_\ell^2 + \text{osc}_\ell^2$ and reliability $\|\mathbf{u} - \mathbf{U}_\ell\| \lesssim \varrho_\ell$, we conclude the proof. \square

Remark. (i) For $p = 1$ and 2D, one may also use nodal interpolation $U_{0,\ell} = I_\ell u_0$ to discretize the Dirichlet data. It is proved in [AFGKMP, Proof of Proposition 5.2] that the limit $U_{0,\infty} = \lim_\ell U_{0,\ell}$ then exists strongly in $H^1(\Gamma)$ and hence also in $H^{1/2}(\Gamma)$.

(ii) For $p = 1$ in 3D and newest vertex bisection, one may also use the L^2 -orthogonal projection $\Pi_\ell : L^2(\Gamma) \rightarrow \mathcal{S}^1(\mathcal{E}_\ell^\Gamma)$. It is proved in [KPP, Theorem 3] that Π_ℓ is H^1 -stable and in [KOP, Lemma 12] that the limit $U_{0,\infty} = \lim_\ell U_{0,\ell}$ therefore exists weakly in $H^1(\Gamma)$ and, by interpolation, hence strongly in $H^{1/2}(\Gamma)$.

(iii) In either case, the claims (3.22)–(3.23) of Theorem 15 remain valid. \square

3.8. Extension to nonlinear transmission problems. We consider the interface problem (3.1) with *nonlinear* $\mathfrak{A} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $\mathfrak{A}\nabla u(x) := \mathfrak{A}(\nabla u(x))$, see e.g. [CS2, AFFKMP]. Under monotonicity (3.5) and Lipschitz continuity (3.6), the coupling formulation (3.3) as well as the discretizations (3.11) and (3.13) still admit unique solutions. Moreover, the *a posteriori* error analysis of Proposition 13 remains valid. Here, note that $\mathfrak{A}\nabla U_\ell$ is \mathcal{T}_ℓ -piecewise Lipschitz continuous so that all occurring terms as e.g. the jump terms (3.15) are well-defined.

While the convergence result of Theorem 15 holds for arbitrary polynomial degrees $p \geq 1$ and $q \geq 0$ in case of linear problems, we did not succeed to prove the same for nonlinear transmission problem. Difficulties are only met in the first step of the proof. In (3.24), the term

$$\sum_{T' \in \mathcal{T}_{\ell+1}} |T'|^{2/d} \|\operatorname{div}(\mathfrak{A}\nabla U_{\ell+1} - \mathfrak{A}\nabla U_\ell)\|_{L^2(T')}^2 \simeq \|h_\ell \operatorname{div}_\ell(\mathfrak{A}\nabla U_{\ell+1} - \mathfrak{A}\nabla U_\ell)\|_{L^2(\Omega)}^2$$

arises, with div_ℓ the \mathcal{T}_ℓ -piecewise divergence operator. However, for lowest-order volume elements $p = 1$, this term vanishes. Thus, our proof also covers the adaptive FEM-BEM coupling for certain nonlinear coupling problems and lowest-order elements $p = 1$, where $\mathfrak{A} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ satisfies (3.5)–(3.6).

For $p \geq 1$ and \mathfrak{A} being linear, we used a scaling argument to bound the latter term in (3.24) by $\|\mathfrak{A}\|_{W^{1,\infty}(\Omega)} \|\nabla U_{\ell+1} - \nabla U_\ell\|_{L^2(\Omega)}$. For $\mathfrak{A} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ being nonlinear, we did neither succeed to find a similar bound (which seems, in general, to be impossible) nor to prove that this term vanishes as $\ell \rightarrow \infty$.

4. NUMERICAL EXPERIMENTS

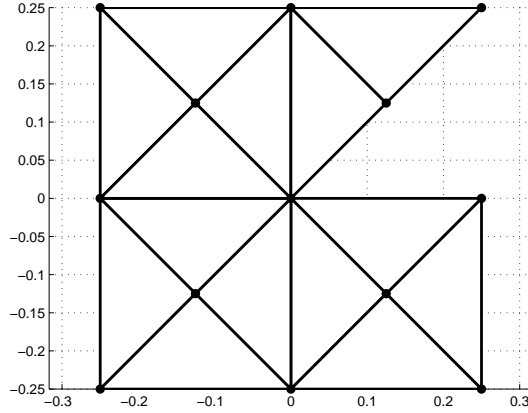


FIGURE 1. Z-shaped domain and initial triangulations $\mathcal{T}_0, \mathcal{E}_0^\Gamma$ with $\#\mathcal{T}_0 = 14$ triangles and $\#\mathcal{E}_0^\Gamma = 10$ boundary elements.

In this section, we present 2D calculations for the perturbed Galerkin discretization (3.13) of the model problem (3.1). We consider different choices of the polynomial degrees p and q for the spaces $\mathcal{S}^p(\mathcal{T}_\ell)$, $\mathcal{P}^q(\mathcal{E}_\ell^\Gamma)$. For ease of implementation and also for stability reasons, we restrict ourselves to the case $\mathcal{E}_\ell^\Gamma = \mathcal{T}_\ell|_\Gamma$, i.e., the boundary mesh is taken as the trace of the

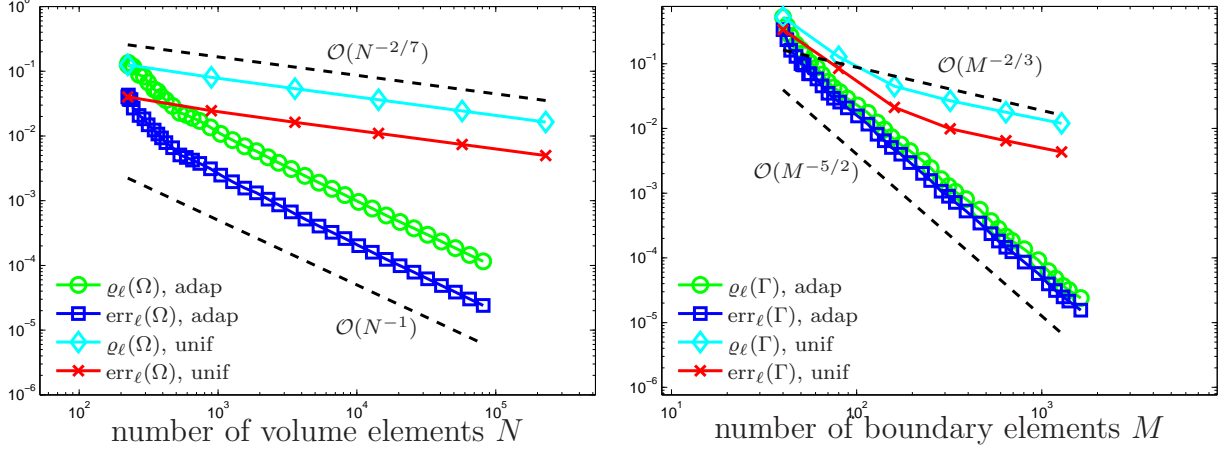


FIGURE 2. $\text{err}_\ell(\Omega)$ and $\varrho_\ell(\Omega)$ vs. $N = \#\mathcal{T}_\ell$ (left) as well as $\text{err}_\ell(\Gamma)$ and $\varrho_\ell(\Gamma)$ vs. $M = \#\mathcal{E}_\ell^\Gamma$ for adaptive and uniform mesh refinement with $p = 2$ and $q = 1$.

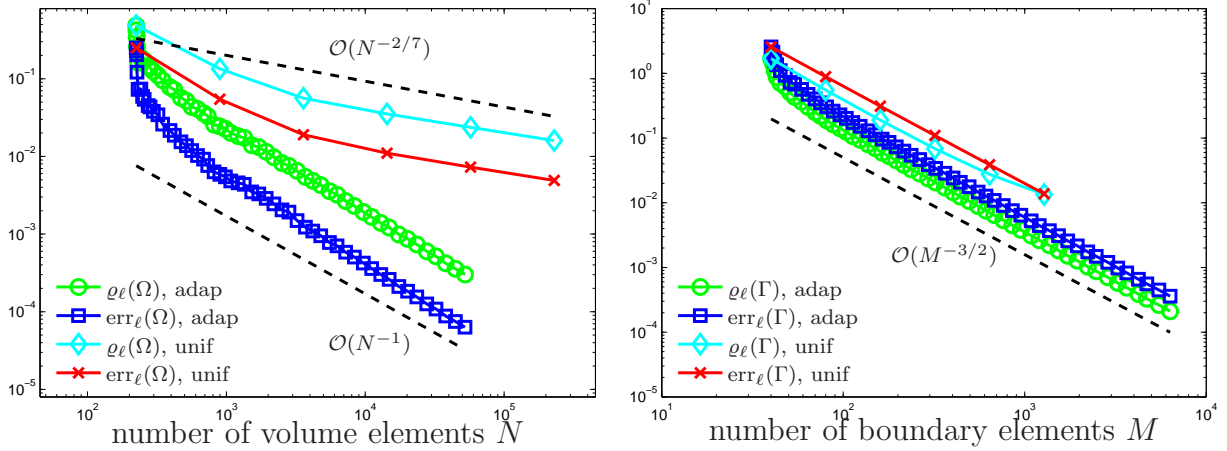


FIGURE 3. $\text{err}_\ell(\Omega)$ and $\varrho_\ell(\Omega)$ vs. $N = \#\mathcal{T}_\ell$ (left) as well as $\text{err}_\ell(\Gamma)$ and $\varrho_\ell(\Gamma)$ vs. $M = \#\mathcal{E}_\ell^\Gamma$ for adaptive and uniform mesh refinement with $p = 2$ and $q = 0$.

volume triangulation. All computations were performed on a 64-BIT Intel(R) Core(TM) i7-3930K Linux work station with 32GB of RAM. For the computation of the discrete boundary integral operators we used the MATLAB BEM-library HILBERT [ABEM], and all systems of linear equations were solved with the MATLAB backslash operator.

We employ the adaptive Algorithm 14 with $\theta = 0.25$ and compare the results with uniform mesh refinement (this can be realized by setting $\theta = 1$ in Algorithm 14). The domain Ω is taken as the Z-shaped domain visualized in Fig. 1. We take $\mathfrak{A} = \text{Id}$ in (3.1) and prescribe the data f , u_0 , ϕ_0 such that the exact solution is given by

$$u^{\text{int}}(r, \varphi) = r^{4/7} \sin\left(\frac{4}{7}\varphi\right),$$

$$u^{\text{ext}}(x, y) = \frac{x + y + 0.25}{|x + \frac{1}{8}|^2 + |y + \frac{1}{8}|^2};$$

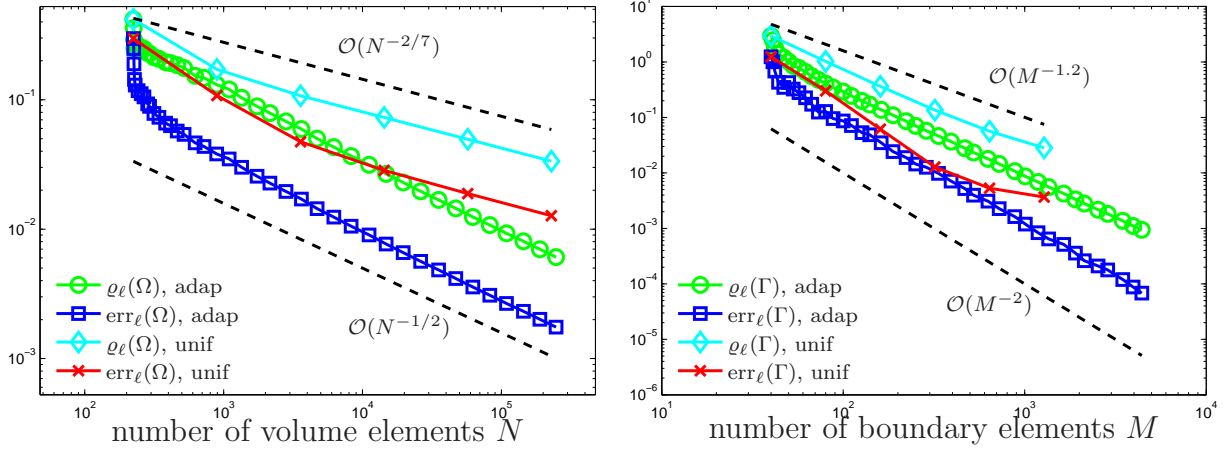


FIGURE 4. $\text{err}_\ell(\Omega)$ and $\varrho_\ell(\Omega)$ vs. $N = \#\mathcal{T}_\ell$ (left) as well as $\text{err}_\ell(\Gamma)$ and $\varrho_\ell(\Gamma)$ vs. $M = \#\mathcal{E}_\ell^\Gamma$ for adaptive and uniform mesh refinement with $p = 1$ and $q = 1$.

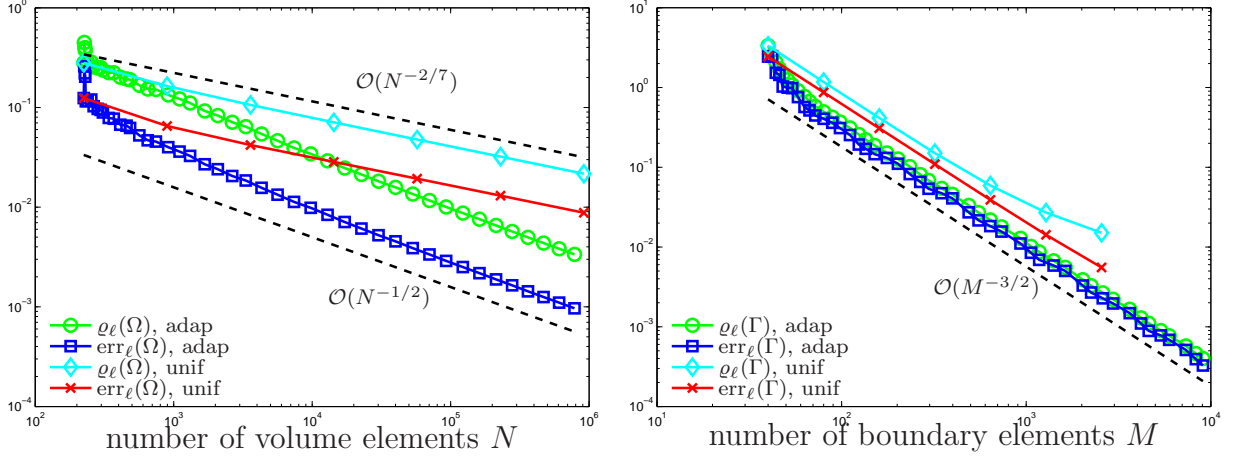


FIGURE 5. $\text{err}_\ell(\Omega)$ and $\varrho_\ell(\Omega)$ vs. $N = \#\mathcal{T}_\ell$ (left) as well as $\text{err}_\ell(\Gamma)$ and $\varrho_\ell(\Gamma)$ vs. $M = \#\mathcal{E}_\ell^\Gamma$ for adaptive and uniform mesh refinement with $p = 1$ and $q = 0$.

the polar coordinates (r, φ) are taken with respect to the origin $(0, 0)$. The prescribed solution u^{int} has the typical singularity at the reentrant corner $(x, y) = (0, 0)$, which leads to a reduced order of convergence $\mathcal{O}(h^{4/7})$ for uniform mesh refinement.

Recall that the *a posteriori* error estimator ϱ_ℓ from (3.18) is split into volume contributions (3.14)–(3.15) and boundary contributions (3.16)–(3.17)

$$\varrho_\ell^2 = \varrho_\ell(\Omega)^2 + \varrho_\ell(\Gamma)^2.$$

Arguing as in [AFP], the Céa-type quasi-optimality produces

$$\begin{aligned} \|\mathbf{u} - \mathbf{U}_\ell\|^2 &= \|u - U_\ell\|_{H^1(\Omega)}^2 + \|\phi - \Phi_\ell\|_{H^{-1/2}(\Gamma)}^2 \lesssim \|u - U_\ell\|_{H^1(\Omega)}^2 + \|h_\ell^{1/2}(\phi - \Phi_\ell)\|_{L^2(\Gamma)}^2 \\ &=: (\text{err}_\ell(\Omega))^2 + (\text{err}_\ell(\Gamma))^2 \end{aligned}$$

and thus provides a computable upper bound for the overall error. We plot the error contribution $\text{err}_\ell(\Omega)$ as well as the corresponding estimator part $\varrho_\ell(\Omega)$ versus the number of volume

elements $N = \#\mathcal{T}_\ell$. We also plot the boundary contributions $\text{err}_\ell(\Gamma)$ and $\varrho_\ell(\Gamma)$ versus the number of boundary elements $M = \#\mathcal{E}_\ell^\Gamma$. We observe convergence rates proportional to $N^{-\alpha}$ resp. $M^{-\beta}$ with some $\alpha, \beta > 0$ for all computed quantities. The optimal convergence rate for the FEM part with $\mathcal{S}^p(\mathcal{T}_\ell)$ is $\alpha = p/2$, whereas the optimal rate for the BEM part with $\mathcal{P}^q(\mathcal{E}_\ell^\Gamma)$ is $\beta = 3/2 + q$. Consequently, the optimal overall convergence rate for the FEM-BEM coupling is proportional to $N^{-\alpha} + M^{-\beta}$.

The use of a uniform mesh refinement leads to suboptimal convergence rates for both the volume and boundary quantities. In particular, we observe the rate $\alpha = 2/7$ independently of the chosen polynomial order p, q , see Figures 2, 3, 4, and 5. We recall that for uniform mesh refinement there holds $N \simeq M^2$ and therefore the overall convergence rate is proportional to $N^{-2/7}$.

In contrast to the uniform approach, the adaptive mesh refinement strategy recovers the optimal overall convergence rate. We observe optimal rates $\alpha = p/2$ and $\beta = 3/2 + q$ for $p = 2, q = 1$ as well as $p = 2, q = 0$ and $p = 1, q = 0$, see Figures 2, 3, and 5. In the case of $p = 1, q = 1$ we observe that the error estimator contribution $\varrho_\ell(\Gamma)$ converges with order $M^{-3/2}$, whereas the error quantity $\text{err}_\ell(\Gamma)$ has some slightly higher convergence rate. Nevertheless, we stress that the optimal overall convergence order is achieved, since our computations also show that $N \simeq M^2$ in the case of $p = 1, q = 1$ and therefore the overall error is dominated by the FEM part, which converges with order $\alpha = 1/2$, see Figure 4.

In conclusion, our numerical experiments underline the fact that the adaptive mesh refinement strategy of Algorithm 14 achieves the optimal convergence rates in the presence of singularities of the exact solution.

Acknowledgement. The research of the authors M. Aurada, M. Feischl, M. Karkulik, and D. Praetorius is supported through the FWF project *Adaptive Boundary Element Method*, funded by the Austrian Science Fund (FWF) under grant P21732. The research of the author T. Führer is additionally supported through the project *Effective Numerical Methods for the Johnson-Nédélec Coupling of FEM and BEM* funded by the innovative projects initiative of Vienna University of Technology.

REFERENCES

- [AO] M. AINSWORTH, J.T. ODEN: *A posteriori error estimation in finite element analysis*, Wiley-Interscience, New-York, 2000.
- [ABEM] M. AURADA, M. EBNER, M. FEISCHL, S. FERRAZ-LEITE, T. FÜHRER, P. GOLDENITS, M. KARKULIK, M. MAYR, D. PRAETORIUS: *HILBERT — A MATLAB implementation of adaptive 2D-BEM*, ASC Report **24/2011**, Institute for Analysis and Scientific Computing, Vienna University of Technology, Wien, 2011, software download at <http://www.asc.tuwien.ac.at/abem/hilbert/>
- [AFFKMP] M. AURADA, M. FEISCHL, T. FÜHRER, M. KARKULIK, J. M. MELENK, D. PRAETORIUS: *Classical FEM-BEM coupling methods: nonlinearities, well-posedness, and adaptivity*, Comp. Mech., online first.
- [AFFKP] M. AURADA, M. FEISCHL, T. FÜHRER, M. KARKULIK, D. PRAETORIUS: *Efficiency and optimality of some weighted-residual error estimator for adaptive 2D boundary element methods*, ASC Report **15/2012**, Institute for Analysis and Scientific Computing, Vienna University of Technology, 2012.
- [AFLP] M. AURADA, S. FERRAZ-LEITE, D. PRAETORIUS: *Estimator reduction and convergence of adaptive BEM*, Appl. Numer. Math. **62** (2012), 787–801.
- [AFP] M. AURADA, M. FEISCHL, D. PRAETORIUS: *Convergence of Some Adaptive FEM-BEM Coupling for elliptic but possibly nonlinear interface problems*, Math. Model. Numer. Anal. **46** (2012), 1147–1173.

- [AFGKMP] M. AURADA, S. FERRAZ-LEITE, P. GOLDENITS, M. KARKULIK, D. PRAETORIUS: *Convergence of adaptive BEM for some mixed boundary value problem*, Appl. Numer. Math. **62** (2012), 226–245.
- [AKP] M. AURADA, M. KARKULIK, D. PRAETORIUS: *Simple error estimators for hypersingular integral equations in adaptive 3D-BEM*, work in progress 2012
- [BS] I. BABUSKA, T. STROUBOULIS: *The finite element method and its reliability*, Numerical Mathematics and Scientific Computation. The Clarendon Press, Oxford University Press, New York, 2001.
- [C1] C. CARSTENSEN: *Efficiency of a posteriori BEM-error estimates for first-kind integral equations on quasi-uniform meshes*, Math. Comp. **65** (1996), 69–84.
- [C2] C. CARSTENSEN: *An a posteriori error estimate for a first-kind integral equation*, Math. Comp. **66** (1997), 139–155.
- [CMS] C. CARSTENSEN, M. MAISCHAK, E. STEPHAN: *A posteriori error estimate and h-adaptive algorithm on surfaces for Symm’s integral equation*, Numer. Math. **90** (2001), 197–213.
- [CMPS] C. CARSTENSEN, M. MAISCHAK, D. PRAETORIUS, E. STEPHAN: *Residual-based a posteriori error estimate for hypersingular equation on surfaces*, Numer. Math. **97** (2004), 397–426.
- [CP1] C. CARSTENSEN, D. PRAETORIUS: *Averaging techniques for the effective numerical solution of Symm’s integral equation of the first kind*, SIAM J. Sci. Comput. **27** (2006), 1226–1260.
- [CP2] C. CARSTENSEN, D. PRAETORIUS: *Averaging techniques for the a posteriori BEM error control for a hypersingular integral equation in two dimensions*, SIAM J. Sci. Comput. **29** (2007), 782–810.
- [CP3] C. CARSTENSEN, D. PRAETORIUS: *Convergence of adaptive boundary element methods*, J. Integral Equations Appl. **24** (2012), 1–23.
- [CS1] C. CARSTENSEN, E. STEPHAN: *A posteriori error estimates for boundary element methods*, Math. Comp. **64** (1995), 483–500.
- [CS2] C. CARSTENSEN, E. STEPHAN: *Adaptive coupling of boundary elements and finite elements*, Math. Model. Numer. Anal. **29** (1995), 779–817.
- [CKNS] J. CASCON, C. KREUZER, R. NOCHETTO, K. SIEBERT: *Quasi-optimal convergence rate for an adaptive finite element method*, SIAM J. Numer. Anal. **46** (2008), 2524–2550.
- [CN] J. CASCON, R. NOCHETTO: *Quasioptimal cardinality of AFEM driven by nonresidual estimators*, IMA J. Numer. Anal. **32** (2012), 1–29.
- [DS] T. DUPONT AND L. R. SCOTT: *Polynomial approximation of functions in Sobolev spaces*, Math. Comp. **34** (1980), 441–463.
- [FKMP] M. FEISCHL, M. KARKULIK, J. M. MELENK, D. PRAETORIUS: *Quasi-optimal convergence rate for an adaptive boundary element method*, ASC Report **28/2011**, Institute for Analysis and Scientific Computing, Vienna University of Technology, 2011.
- [FPP] M. FEISCHL, M. PAGE, D. PRAETORIUS: *Convergence and quasi-optimality of adaptive FEM with inhomogeneous Dirichlet data*, ASC Report **34/2010**, Institute for Analysis and Scientific Computing, Vienna University of Technology, 2010.
- [FOP] S. FERRAZ-LEITE, C. ORTNER, D. PRAETORIUS: *Convergence of simple adaptive Galerkin schemes based on $h - h/2$ error estimators*, Numer. Math. **116** (2010), 291–316.
- [FP] S. FERRAZ-LEITE, D. PRAETORIUS: *Simple a posteriori error estimators for the h-version of the boundary element method*, Computing **83** (2008), 135–162.
- [G] T. GANTUMUR: *Adaptive boundary element methods with convergence rates*, Preprint, **arXiv:1107.0524v2**, 2011.
- [EG] E.H. GEORGOULIS: *Inverse-type estimates on hp-finite element spaces and applications*, Math. Comp. **77** (2007), 201–219.
- [GHS] I. GRAHAM, W. HACKBUSCH, S. SAUTER: *Finite elements on degenerate meshes: Inverse-type inequalities and applications*, IMA J. Numer. Anal. **25** (2005), 379–407.
- [HW] G. HSIAO, W. WENDLAND: *Boundary integral equations*, Springer, Berlin, 2008.
- [KOP] M. KARKULIK, G. OF, AND D. PRAETORIUS: *Convergence of adaptive 3D-BEM for weakly singular integral equations based on isotropic mesh refinement*, ASC Report **20/2012**, Institute for Analysis and Scientific Computing, Vienna University of Technology, 2012.

- [KPP] M. KARKULIK, D. PAVLICEK, D. PRAETORIUS: *On 2D newest vertex bisection: Optimality of mesh-closure and H^1 -stability of L_2 -projection*, ASC Report **10/2012**, Institute for Analysis and Scientific Computing, Vienna University of Technology, 2012.
- [K] M. KARKULIK: *Zur Konvergenz und Quasioptimalität adaptiver Randelementmethoden*, PhD-thesis (in German), Institute for Analysis and Scientific Computing, Vienna University of Technology, 2012.
- [M] W. MCLEAN: *Strongly elliptic systems and boundary integral equations*, Cambridge University Press, Cambridge, 2000.
- [MO] C. B. MORREY, JR.: *Multiple integrals in the calculus of variations*. Classics in Mathematics. Springer, Berlin, 2008.
- [RS] S. RJASANOW, O. STEINBACH: *The fast solution of boundary integral equations*, Springer, New York, 2007.
- [SS] S. SAUTER, C. SCHWAB: *Boundary element methods*, Springer, Berlin, 2011.
- [SZ] L. R. SCOTT, S. ZHANG: *Finite element interpolation of nonsmooth functions satisfying boundary conditions*, Math. Comp. **54** (1990), 483–493.
- [S] O. STEINBACH: *Numerical approximation methods for elliptic boundary value problems: Finite and boundary elements*, Springer, New York, 2008.
- [St] R. STEVENSON: *The completion of locally refined simplicial partitions created by bisection*, Math. Comp. **77** (2008), 227–241.
- [Ver84] G. VERCHOTA: *Layer potentials and regularity for the Dirichlet problem for Laplace’s equation in Lipschitz domains*, J. Funct. Anal. **59** (1984), 572–611.
- [V] R. VERFÜRTH: *A review of a posteriori error estimation and adaptive mesh-refinement techniques*, Teubner, Stuttgart, 1996.

INSTITUTE FOR ANALYSIS AND SCIENTIFIC COMPUTING, VIENNA UNIVERSITY OF TECHNOLOGY, WIEDNER HAUPTSTRASSE 8-10, A-1040 WIEN, AUSTRIA
E-mail address: {Michael.Feischl,Thomas.Fuehrer}@tuwien.ac.at
E-mail address: {Melenk,Dirk.Praetorius}@tuwien.ac.at
E-mail address: Michael.Karkulik@tuwien.ac.at (corresponding author)